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MATHEMATICS

On Some Spaces of Functions and Distributions. III. Spaces of Vector-valued M-integrable Distributions

by

J. MUSIELAK

Presented by W. ORLICZ on August 30, 1961

1. Let $M(u)$ be an even, continuous, convex, nonnegative function vanishing only at 0, $M(u)/u \to 0$ as $u \to 0$, $M(u)/u \to \infty$ as $u \to \infty$ and let $N(u)$ be the function complementary to $M(u)$ in the sense of Young. Applying the notation of [5] and denoting $\mathcal{D}_N = \{\varphi \in \mathcal{E}\colon \int N(k_p D^p \varphi(x))\,dx < \infty$ for a $k_p > 0$ dependent on $\varphi\}$, where the integral is taken over the whole n-dimensional space R^n, and defining the topology in $\mathcal{D}_N$ by a countable number of seminorms $\|\varphi\|_{Np} = \inf\{\varepsilon > 0\colon \int N(\varepsilon^{-1} D^p \varphi(x))\,dx \leqslant 1\}$, $\mathcal{D}_N$ is a B_0-space; if $N(u)$ satisfies the well-known condition (Δ_2) for small u, the set $\mathcal{D}$ of infinitely differentiable functions of compact support is dense in $\mathcal{D}_N$ ([3], 2.2 (c)). The dual of $\mathcal{D}_N$ is denoted by $\mathcal{D}'_M$. The Orlicz space $\mathcal{L}^*_M$ is linearly isomorphic with a part of $\mathcal{D}'_M$ and the topology defined in this part by $\mathcal{L}^*_M$ is finer than the topology induced by $\mathcal{D}'_M$ ([3], 2.3 (b)). Assuming $M(u)$ and $N(u)$ to satisfy the condition (Δ_2) for all u, $\mathcal{D}$ is dense in $\mathcal{D}'_M$ ([4], Lemma 3); hence in this case $\mathcal{D}'_M$ is a normal space of distributions (for the definition see [7], p. 7). In this note we shall always assume that the functions $M(u)$ and $N(u)$ satisfy the condition (Δ_2) for all u.

2. Let E be a locally convex Hausdorff linear topological space and let $\mathcal{H}$ be any space of distributions, i.e. a locally convex Hausdorff linear topological space linearly isomorphic with a part of the space $\mathcal{D}'$ of all distributions with a topology finer than in $\mathcal{D}'$. L. Schwartz ([6], 20, and [7], p. 52) introduced the *space of vector-valued distributions corresponding to the space* $\mathcal{H}$, defined as $\mathcal{H}(E) = \mathcal{H}\varepsilon E = \mathcal{L}_\varepsilon(\mathcal{H}'_c; E)$. Here $\mathcal{L}_\varepsilon(A'; B)$ is the space of continuous linear operations from A' to B with the topology of uniform convergence on equicontinuous parts of A', and A'_c is the dual of A provided with the topology of uniform convergence on convex, compact, stable subsets of A ([7], p. 18 and p. 34, Coroll. 2; cf. also [1], III. 3.5). Evidently, since $\mathcal{D}$ is a Montel space, if $\mathcal{H} = \mathcal{D}'$ then we have $\mathcal{D}'(E) = L_\varepsilon(\mathcal{D}_c; E) = \mathcal{L}_\varepsilon(\mathcal{D}; E) = \mathcal{L}_b(\mathcal{D}; E)$, where b means the topology of uniform convergence on bounded sets. Moreover, $\mathcal{H}(E)$ is linearly isomorphic with a part of $\mathcal{D}'(E)$, the topology induced in this part by $\mathcal{H}(E)$ being finer than that induced by $\mathcal{D}'(E)$ ([7], p. 52). The elements of $\mathcal{D}'(E)$ are called *vector-valued distributions*

with value in E and are denoted by $\vec{T}$. Besides $\mathcal{H}(E)$, the space $\overline{\mathcal{H}}(E) = \mathcal{L}_\varepsilon(\mathcal{H}'; E)$ may be also considered. The above formula shows that in case when $\mathcal{H} = \mathcal{D}'$, we have $\overline{\mathcal{D}}'(E) = \mathcal{D}'(E)$.

3. Now, put $\mathcal{H} = \mathcal{D}'_M$. Since the dual of $\mathcal{D}'_M$ is $\mathcal{D}_N$([4], Lemma 1), we have $\mathcal{D}'_M(E) = \mathcal{L}_\varepsilon(\mathcal{D}_{Nc}; E)$. The same result is obtained if we take $\mathcal{H} = D'_{Mc}$, for the dual of $\mathcal{D}'_{Mc}$ is also $\mathcal{D}_N$([7], p. 16, and [1], IV. 2.3, Theorem 2 and Corollary); thus $D'_{Mc}(E) = \mathcal{D}'_M(E)$. The space $\mathcal{D}_N$ being a normal space of distributions, $\mathcal{H} = \mathcal{D}_{Mc}$ possesses the following property (ε) ([7], Prop. 14): *Let E be any locally convex Hausdroff quasi-complete linear topological space, and let $\vec{T}$ be any vector-valued distribution such that the range of the adjoint $\vec{T}^*$ of $\vec{T}$ (an operation from E'_c to $\mathcal{D}'$) is contained in $\mathcal{H}$. Then $\vec{T}^*$ is continuous as an operation from E'_c to $\mathcal{H}$ (where $\mathcal{H}$ is provided with its initial topology)*. Considering now the space $\overline{\mathcal{D}}'_M(E) = \mathcal{L}_\varepsilon(\mathcal{D}_N; E) = \mathcal{L}_\varepsilon(\mathcal{D}_{Nk}; E)$, where k denotes the Mackey's topology (cf. [1], IV. 2. 3, Prop. 5), let us note that $\mathcal{D}'_M(E)$ is a linear topological subspace of $\overline{\mathcal{D}}'_M(E)$. We shall call distributions belonging to $\mathcal{D}'_M(E)$ resp. $\overline{\mathcal{D}}'_M(E)$, *cM-integrable resp. M-integrable, E-valued distributions*. In case when E is the space of real numbers R^1 we have $\overline{\mathcal{D}}'_M(E) = \mathcal{D}'_M$, for equicontinuous parts of $\mathcal{D}_N$ are bounded in $\mathcal{D}_N$ ([1], III. 3, 6. Prop. 7) and conversely.

In addition to the considerations of [4] we may observe here that taking two reflexive spaces of distributions $\mathcal{H}$ and $\mathcal{K}$, the space $\overline{\mathcal{K}}'(\mathcal{H})$ consists of all adjoints of the operations belonging to $\overline{\mathcal{H}}(\mathcal{K}')$. Take now two functions $M_1(u)$, $M_2(u)$ and their complementary functions $N_1(u)$, $N_2(u)$ satisfying the condition (Δ_2) for all u. In the special case when $\mathcal{H} = \mathcal{L}^*_{M_2}$ and $\mathcal{K} = \mathcal{D}'_{M_1}$ (these spaces are reflexive, cf. [4], Lemma 1) we obtain the spaces $\overline{\mathcal{L}}^*_{M_2}(\mathcal{D}_{N_1}) = \mathcal{L}_\varepsilon(\mathcal{L}^*_{N_2}; \mathcal{D}_{N_1})$ and $\overline{\mathcal{D}}_{N_1}(\mathcal{L}^*_{M_2}) = \mathcal{L}_\varepsilon(\mathcal{D}'_{M_1}; \mathcal{L}^*_{M_2})$. If a kernel $k(x, y)$ satisfies the assumptions of Theorem 1 in [4] and if $K\varphi(x) = \int k(x, y)\,\varphi(y)\,dy$ and K^* is the adjoint of K (see [4], (1) and (2)), then $K \in \overline{\mathcal{L}}^*_{M_2}(\mathcal{D}_{N_1})$ and $K^* \in \overline{\mathcal{D}}_{N_1}(\mathcal{L}^*_{M_2})$. Now, let us consider the spaces $\mathcal{H}(\mathcal{K})'$ and $\mathcal{K}'(\mathcal{H})$. By [7], p. 34, Coroll. 2, we have the isomorphisms $\mathcal{H}(\mathcal{K}') = \mathcal{L}_\varepsilon(\mathcal{H}'_c; \mathcal{K}') = \mathcal{L}_\varepsilon(\mathcal{K}_c; \mathcal{H}) = \mathcal{K}'(\mathcal{H})$; in particular, the spaces $\mathcal{L}^*_{M_2}(\mathcal{D}_{N_1})$ and $\mathcal{D}_{N_1}(\mathcal{L}^*_{M_2})$ are isomorphic. Obviously, $\mathcal{L}^*_{M_2}(\mathcal{D}_{N_1})$ and $\mathcal{D}_{N_1}(\mathcal{L}^*_{M_2})$ are linear topological subspaces of $\overline{\mathcal{L}}^*_{M_2}(\mathcal{D}_{N_1})$ and $\overline{\mathcal{D}}_{N_1}(\mathcal{L}^*_{M_2})$, respectively. The author will turn to the question of continuity and compactness of the integral operators K and K^* in the topologies c in another note.

4. We assume now that E is a Banach space and we denote by $\mathcal{L}^*_M[E]$ the Orlicz space of all vector-valued strongly measurable functions $f(x)$ defined over the n-dimensional space R^n with values in E, which are M-Bochner-integrable, i.e. $\int M(k\,\|f(x)\|)\,dx < \infty$ for a $k > 0$ dependent of f (in the case of powers, cf. e. g. [2], III. 3). Defining the norm in $\mathcal{L}^*_M[E]$ as $\|f\|_M = \inf\{\varepsilon > 0 : \int M(\varepsilon^{-1}\|f(x)\|)\,dx \leqslant 1\}$, $\mathcal{L}^*_M[E]$ is a Banach space. Take now an arbitrary function $f \in \mathcal{L}^*_M[E]$ and denote $\vec{T}_f(\varphi) = \int f(x)\,\varphi(x)\,dx$ for any $\varphi \in \mathcal{D}_N$. It is easily seen that $\vec{T}_f$ is a linear continuous operation from $\mathcal{D}_N$ to E, i.e. *$\vec{T}_f$ is an M-integrable E-valued distribution*. Indeed, the vector-valued function $f(x)\,\varphi(x)$ is Bochner-integrable and since (*)

$\|\vec{T}_f(\varphi)\| \leqslant \|f\|_M \|\varphi\|_{No}$, $\vec{T}_f$ is continuous. Denoting by $A \subset \overline{\mathcal{D}}'_M(E)$ the set of all $\vec{T}_f$ with $f \in \mathcal{L}^*_M[E]$ we shall show now that the topology induced in A by $\mathcal{L}^*_M[E]$ is finer than the topology induced by $\overline{\mathcal{D}}'_M(E)$. Indeed, since equicontinuous sets are bounded, it suffices to show that the topology induced in A by $\mathcal{L}^*_M[E]$ is finer than the topology induced in A by $\mathcal{L}_b(\mathcal{D}_N; E)$. However, given any neighbourhood of zero in $\mathcal{L}_b(\mathcal{D}_N; E)$ of the form $U = \{\vec{T} \in \mathcal{L}(\mathcal{D}_N; E)\colon \|\vec{T}(\varphi)\| \leqslant 1 \text{ for } \varphi \in B\}$, where B is a bounded set in $\mathcal{D}_N$, we have $\|\varphi\|_{No} \leqslant \lambda_0$ for a $\lambda_0 > 0$ and taking $\|f\|_M \leqslant \lambda_0^{-1}$ we get by (*), $\|\vec{T}_f(\varphi)\| \leqslant 1$, i.e. $\vec{T}_f \in U$. Thus we have obtained the following result being a generalization of [3], 2.3 (b): *The space $\mathcal{L}^*_M[E]$ is linearly isomorphic with a part of $\overline{\mathcal{D}}'_M(E)$ and the topology induced in this part by $\mathcal{L}^*_M[E]$ is finer than the topology induced by $\overline{\mathcal{D}}'_M(E)$.*

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REFERENCES

[1] N. Bourbaki, *Espaces vectoriels topologiques* I, II, Paris, 1953, 1954.

[2] N. Dunford, J. T. Schwartz, *Linear operators* I, New York, 1958.

[3] J. Musielak, *On some spaces of functions and distributions* (I), *Spaces $\mathcal{D}_M$ and $\mathcal{D}'_M$*, Studia Mathem. (in press).

[4] — , *On some spaces of functions and distributions (II)*, *Integral transforms in $\mathcal{D}_M$ and $\mathcal{D}'_M$*, Studia Mathem. (in press).

[5] L. Schwartz, *Théorie des distributions* I, II, Paris, 1950, 1951.

[6] — , Séminaire 1953/54, Paris, 1954.

[7] — , *Théorie des distributions à valeurs vectorielles*, Ann. de l'Institut Fourier **7** (1957), 1—141.

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MATHEMATICS

More About (K, φ)-summability

by

R. TABERSKI

Presented by W. ORLICZ on August 30, 1961

1. In the present paper we write $|C^q, 1| \sum_{k=0}^{\infty} u_k = s$ when the series $\sum_{k=0}^{\infty} u_k$ with partial sums s_n is strongly summable with exponent q to s, that is $\sum_{\nu=0}^{n} |s_\nu - s|^q = o(n)$. By $\omega_n(x)$ we denote an orthonormal system in the interval $\langle a, b\rangle$. The Fourier series (in real form) of the 2π-periodic Lebesgue integrable function f is denoted by $S[f]$ or $\sum_{k=0}^{\infty} v_k(x)$. The Fourier-Lagrange series of 2π-periodic continuous f with respect to the weight $d\mu_n(t) = \frac{2\pi}{2n+1}\, dE\left(\frac{2n+1}{2\pi}\, t\right)$ $(-\infty < t < \infty)$ will be denoted by $\sum_{k=0}^{n} v_k^*(x)$ ([5], pp. 4—7). Other notations, which are not explained here, are as in [3], §§ 1,5.

2. First the relations between summabilities $|C^q, 1|$ and (K, φ) (see [3], § 1) and next a theorem on (K, φ)-summability of differentiated series $S^{(r)}[f]$ will be given.

2.1. *If $\varphi(x)$ is of bounded variation in $\langle 0, 1\rangle$, then $|C^1, 1| \sum_{k=0}^{\infty} u_k = s$ implies $(K, \varphi) \sum_{k=0}^{\infty} u_k = s$ $(|s| < \infty)$.*

To obtain this, we make use of the inequalities concerning $\lambda_k^{(n)} = \varrho_k^{(n)} - \varrho_{k+1}^{(n)}$, given in [3], § 2.1, and we apply the following analogue of the well-known theorem of Faddeev for singular integrals (cf. [4], § 9).

2.2. *A necessary and sufficient condition for*

$$\lim_{n\to\infty} \sum_{k=0}^{\infty} a_k b_k^{(n)} = g \qquad (|g| < \infty),$$

whenever

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n} |a_k - g| = 0,$$

is that

$$\lim_{n\to\infty} \sum_{k=r}^{\infty} b_k^{(n)} = 1 \quad (r = 0, 1, 2, \ldots) \quad \text{and} \quad \overline{\lim_{n\to\infty}} \sum_{k=0}^{\infty} \sup_{k\geq l} |b_k^{(n)}| < \infty.$$

2.3. *Let* $\varphi(x)$ *be as in* 2.1 *and let* $\sum_{k=0}^{\infty} c_k^2 < \infty$.

Then the orthogonal series $\sum_{k=0}^{\infty} c_k \omega_k(x)$ *is* $|C^2, 1|$*-summable to* $s(x)$, *i. e.* $|C^2, 1| \sum_{k=0}^{\infty} c_k \omega_k(x) = s(x)$, *at almost all* $x \in \langle a, b\rangle$ *at which*

$$(K, \varphi) \sum_{k=0}^{\infty} c_k \omega_k(x) = s(x).$$

Proof. Write

$$s_n(x) = \sum_{k=0}^{n} c_k \omega_k(x), \qquad \tau_n(x) = \sum_{k=0}^{n} \varrho_k^{(n)} c_k \omega_k(x),$$

$$\Delta_n(x) = s_n(x) - \tau_n(x) = \sum_{k=0}^{n} (1 - \varrho_k^{(n)})\, c_k \omega_k(x).$$

Since ([3], § 3.1]

$$1 - \varrho_k^{(n)} = \frac{1}{2A_n}\left\{ \sum_{\nu=0}^{n-k+2} \left[\varphi\left(\frac{\nu}{n+2}\right) - \varphi\left(\frac{\nu+k}{n+2}\right)\right]^2 + \right.$$

$$\left. + \sum_{\nu=0}^{k-1} \left[\varphi^2\left(\frac{\nu}{n+2}\right) + \varphi^2\left(1 - \frac{\nu}{n+2}\right)\right]\right\} \leqslant \frac{Qk}{n+1} \quad (k, n = 0, 1, 2, \ldots, \quad Q = \text{const}),$$

we have

$$\sum_{n=1}^{\infty} \frac{1}{n} \int_a^b \Delta_n^2(x)\, dx = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=0}^{n} (1 - \varrho_k^{(n)})^2\, c_k^2 = O(1) \sum_{k=0}^{\infty} c_k^2 < \infty,$$

whence

$$\sum_{n=1}^{\infty} \frac{\Delta_n^2(x)}{n} = \sum_{n=1}^{\infty} \frac{(s_n(x) - \tau_n(x))^2}{n} < \infty$$

almost everywhere in $\langle a, b\rangle$. Now, we proceed as in the proof of the theorem (7.7) of [5], pp. 183—4.

2.4. *Let* $\varphi \in \mathrm{Lip}_M^{(p)} p$ $(p = 1, 2, \ldots, m)$ *in* $\langle 0, 1\rangle$ *and let* $\varphi(t) = O(t^{(m-2)/2})$, $\varphi(1 - t) = O(t^{(m-2)/2})$ *as* $t \to 0 + (m \geqslant 2)$.

Then, if the symmetric derivative $f_{(r)}(x_0)$ *exists* ([5], p. 59—60), *the series* $S^{(r)}[f]$ *is* (K, φ)*-summable at* x_0 *to* $f_{(r)}(x_0)$, *provided* $r < m - 1$, *i.e.*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \varrho_k^{(n)} v_k^{(r)}(x_0) = f_{(r)}(x_0) \qquad (r < m - 1).$$

The last statement follows at once from 2.3 of [3] and (1.7) of [5], p. 60.

3. Now we shall deduce an asymptotic formula concerning the approximation to functions $f \in \mathrm{Lip}_1^{(2)} \alpha$ by trigonometric Korovkin's polynomials

$$I_n(x; f) = \sum_{k=0}^{n} \varrho_k^{(n)} v_k(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \left[\frac{1}{2} + \sum_{k=1}^{n} \varrho_k^{(n)} \cos k(t - x)\right] dt$$

in which the factors $\varrho_k^{(n)}$ are generated by the function $\varphi(x) = \sin \pi x$ ([3], §§ 1, 5).

3.1. *The following formula holds*

$$S_n(\alpha) = \frac{M(\alpha)\,\Gamma(\alpha)\sin\frac{\alpha\pi}{2}}{n^\alpha} + \begin{cases} O\left(\frac{1}{n^{\alpha+1}}\right) & \text{when } 0 < \alpha \leqslant 1, \\ O\left(\frac{\ln n}{n^2}\right) & \text{when } 1 < \alpha < 2, \end{cases}$$

where

$$M(\alpha) = \frac{1}{\pi}\int_0^\pi \left[\left(\frac{\pi}{t}\right)^\alpha - \left(\frac{\pi}{t}\right)^{\alpha-1}\right]\sin t\,dt.$$

Proof. As known ([1], p. 145),

$$S_n(\alpha) = \frac{2^\alpha}{\pi}\Gamma(\alpha)\sin\frac{\alpha\pi}{2}\sum_{k=0}^{n}\frac{\lambda_k^{(n)}}{(2k+1)^\alpha} + O\left(\sum_{k=0}^{n}\frac{|\lambda_k^{(n)}|}{(2k+1)^2}\right)$$

for $0 < \alpha \leqslant 2$, and, if $\alpha = 1$, the correction-term in the last relation can be replaced by $O(\lambda_0^{(n)}) = O(1 - \varrho_1^{(n)}) = O\left(\frac{1}{n^2}\right)$ (see [3], § 5.1). Since ([3], § 5.2)

$$\varrho_k^{(n)} = \left[\frac{n-k+3}{2(n+2)}\sin\frac{(k+1)\pi}{n+2} - \frac{n-k+1}{2(n+2)}\sin\frac{(k-1)\pi}{n+2}\right]\operatorname{cosec}\frac{\pi}{n+2}$$

for $0 \leqslant k \leqslant n+1$, we have

$$\lambda_k^{(n)} = \left[\frac{2(n-k+1)}{n+2}\cos\frac{\pi}{2(n+2)}\sin\frac{(2k+1)\pi}{2(n+2)} + \frac{1}{n+2}\sin\frac{(k+1)\pi}{n+2}\right]\tan\frac{\pi}{2(n+2)} > 0$$

for $0 \leqslant k \leqslant n$. Hence,

$$\sum_{k=0}^{n}\frac{\lambda_k^{(n)}}{(2k+1)^\alpha} = \left[\frac{2n+3}{n+2}J_n(\alpha) - \frac{1}{n+2}J_n(\alpha-1)\right]\sin\frac{\pi}{2(n+2)} + O\left(\frac{1}{n^2}J_n(\alpha)\right),$$

where

$$J_n(\beta) = \sum_{k=0}^{n}\frac{1}{(2k+1)^\beta}\sin\frac{(2k+1)\pi}{2(n+2)} =$$

$$= \frac{1}{2^\beta}\left(\frac{\pi}{n+2}\right)^{\beta-1}\sum_{k=0}^{n}\frac{\pi}{n+2}\left[\frac{2(n+2)}{(2k+1)\pi}\right]^\beta\sin\frac{(2k+1)\pi}{2(n+2)} \qquad (-1 < \beta \leqslant 2).$$

Denote the last sum by $P_n(\beta)$.

It is easily seen that

$$P_n(\beta) = \int_0^\pi \frac{\sin t}{t^\beta}\,dt + \begin{cases} O\left(\frac{1}{n}\right) & \text{when } -1 < \beta \leqslant 1, \\ O\left(\frac{1}{n^{2-\beta}}\right) & \text{when } 1 < \beta < 2, \end{cases}$$

and

$$P_n(2) = O(\ln n);$$

whence, by simple calculation, we get

$$\sum_{k=0}^{n} \frac{\lambda_k^{(n)}}{(2k+1)^\alpha} = \frac{\pi}{(2n)^\alpha} M(\alpha) + \begin{cases} O\left(\frac{1}{n^{\alpha+1}}\right) & \text{if } 0 < \alpha \leqslant 1, \\ O\left(\frac{1}{n^2}\right) & \text{if } 1 < \alpha < 2, \end{cases}$$

and our asymptotic formula follows (see [3], § 5.2).

We still remark that, in view of 2.4,

$$L_n'(x_0; f) \to f_{(1)}(x_0) \quad \text{and} \quad L_n''(x_1; f) \to f_{(2)}(x_1) \quad (\text{as } n \to \infty),$$

provided the derivatives at right exist.

4. Write

$$L_n^*(x; f) = \sum_{k=0}^{n} \varrho_k^{(n)} v_k^*(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t)\, \psi_n(t-x)\, d\mu_n(t),$$

where $\psi_n(t) = \frac{1}{2} + \sum_{k=1}^{n} \varrho_k^{(n)} \cos kt$, $\varrho_k^{(n)}$ are as in § 3, and

$$S_n^*(\alpha) = \sup_{f \in \mathrm{Lip}_1^{(1)}\alpha} \{\max_{-\pi \leqslant x \leqslant \pi} |L_n^*(x; f) - f(x)|\}.$$

4.1. *The following estimate is true*:

$$S_n^*(\alpha) \leqslant \frac{\pi^{2\alpha}}{(2n+4)^\alpha} \quad \text{for } 0 < \alpha \leqslant 1, \quad n = 0, 1, 2, \ldots.$$

Proof. If $f \in \mathrm{Lip}_1^{(1)}\alpha$ $(0 < \alpha \leqslant 1)$, $-\pi \leqslant x \leqslant \pi$, then

$$|L_n^*(x; f) - f(x)| = \frac{1}{\pi} \left| \int_{-\pi}^{\pi} [f(t+x) - f(x)]\, \psi_n(t)\, d\mu_n(t+x) \right| \leqslant$$

$$\leqslant \frac{2^\alpha}{\pi} \int_{-\pi}^{\pi} \left| \frac{t}{2} \right|^\alpha \psi_n(t)\, d\mu_n(t+x) \leqslant \pi^{\alpha-1} \int_{-\pi}^{\pi} \left| \sin \frac{t}{2} \right|^\alpha \psi_n(t)\, d\mu_n(t+x) \leqslant$$

$$\leqslant \pi^{\alpha-1} \left[\int_{-\pi}^{\pi} \sin^2 \frac{t}{2}\, \psi_n(t)\, d\mu_n(t+x) \right]^{\alpha/2} \left[\int_{-\pi}^{\pi} \psi_n(t)\, d\mu_n(t+x) \right]^{1-\alpha/2} \leqslant \frac{\pi^{2\alpha}}{(2n+4)^\alpha}$$

(cf. [3], § 5.2). The proof is thus concluded.

It may also easily be shown that for any $f \in C_{2\pi}$ with modulus of continuity $\omega(\delta)$,

$$\max_{-\pi \leqslant x \leqslant \pi} |L_n^*(x; f) - f(x)| \leqslant 6\omega\left(\frac{1}{n}\right).$$

5. Finally, we shall give two simple results for general positive operators $L_n(x;f)$ and $L_n^*(x;f)$ $(f\in C_{2\pi})$ defined in §§ 3,4. Write

$$p_n=\frac{2}{\pi}\sum_{k=0}^{n}\frac{\varrho_k^{(n)}-\varrho_{k+1}^{(n)}}{2k+1}\qquad(\varrho_{n+1}^{(n)}=0).\quad q_n=1-\varrho_1^{(n)},\quad r_n=3-4\varrho_1^{(n)}+\varrho_2^{(n)}.$$

5.1. *If* $p_n=o(1)$, $q_n=o(p_n)$, *and if the finite limit*

$$\lim_{h\to 0+}\frac{f(x_0-h)-2f(x_0)+f(x_0+h)}{h}=f^{\{1\}}(x_0)$$

exists, then

$$L_n(x_0;f)=f(x_0)+p_nf^{\{1\}}(x_0)+o(p_n).$$

To prove this, it is sufficient to observe that all conditions of theorem 11 in [2] are satisfied. Indeed,

$$c_n=\frac{1}{\pi}\int_0^{\pi}2\sin\frac{t}{2}\,\psi_n(t)\,dt=\frac{1}{\pi}\int_0^{\pi}\sum_{k=0}^{n}[\varrho_k^{(n)}-\varrho_{k+1}^{(n)}]\sin(2k+1)\frac{t}{2}\,dt=p_n=o(1).$$

and, if $0<\delta\leqslant\pi$,

$$d_n=\frac{1}{\pi}\int_{\delta}^{\pi}2\sin\frac{t}{2}\,\psi_n(t)\,dt\leqslant\frac{2}{\pi}\int_{\delta}^{\pi}\psi_n(t)\,dt\leqslant$$

$$\leqslant\frac{2}{\pi(1-\cos\delta)}\int_{\delta}^{\pi}(1-\cos t)\,\psi_n(t)\,dt\leqslant\frac{1-\varrho_1^{(n)}}{1-\cos\delta}=o(c_n).$$

5.2. *Let* $q_n=o(1)$, $r_n=o(q_n)$, *and let the finite second derivative* $f''(x_0)$ *exist. Then*

$$L_n^*(x_0;f)=f(x_0)+q_nf''(x_0)+o(q_n).$$

Proof. Since

$$f(t+x_0)=f(x_0)+f'(x_0)\sin t+\frac{1}{2}f''(x_0)\left(2\sin\frac{t}{2}\right)^2+\gamma(t)\left(2\sin\frac{t}{2}\right)^2,$$

where $\gamma(t)$ is continuous in $\langle-\pi,\pi\rangle$ and $\gamma(t)\to 0$ as $t\to 0$, we have

$$L_n^*(x_0;f)-f(x_0)=f'(x_0)\frac{1}{\pi}\int_{-\pi}^{\pi}\sin t\,\psi_n(t)\,d\mu_n(t+x_0)+$$

$$+f''(x_0)\frac{1}{\pi}\int_{-\pi}^{\pi}2\sin^2\frac{t}{2}\,\psi_n(t)\,d\mu_n(t+x_0)+$$

$$+\frac{2}{\pi}\int_{-\pi}^{\pi}\gamma(t)\,2\sin^2\frac{t}{2}\,\psi_n(t)\,d\mu_n(t+x_0)=f'(x_0)\,G_n+f''(x_0)\,H_n+2J_n.$$

Obviously ([5], p. 7), $G_n = 0$ and

$$H_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (1 - \cos t)\, \psi_n(t)\, d\mu_n(t + x_0) = 1 - \varrho_1^{(n)}.$$

Moreover, $J_n = o(1 - \varrho_1^{(n)})$ (see [3], § 5.1), and our result is established.

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REFERENCES

[1] L. I. Bausov, *On the approximation of Z_α-class' functions by positive methods for summation of Fourier series* [in Russian], Usp. Mat. Nauk, **16** (1961), No. 3 (99), 143—149.

[2] R. Taberski, *On singular integrals*, Ann. Polon. Math. **4** (1958), 249—268.

[3] — , *Some properties of (K, φ)-summability*, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 659—666.

[4] A. Wilansky u. K. Zeller, *Abschnittsbeschränkte Matrixtransformationen; starke Limitierbarkeit*, Math. Zeitschr., **64** (1956), 258—269.

[5] A. Zygmund, *Trigonometric series*, II, Cambridge, 1959.

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MATHEMATICS

An Iterative Method for the Eigenproblem of Linear Bounded Operators and Matrices

by

M. ALTMAN

Presented by W. ORLICZ on August 31, 1961

This paper contains an iterative method of computing eigenvalues and corresponding eigenvectors of self-adjoint linear bounded operators in Hilbert space. The advantage of this method is that starting with an arbitrary vector we get an eigenvalue and the corresponding eigenvector. Thus, the approximate eigenvalue and the corresponding eigenvector depend on the starting vector. Hence, in general, we obtain different eigenvalues and the corresponding eigenvectors by starting with different initial vectors. An application of the method to the eigenproblem of real symmetric matrices is also given. The method in this case is suitable for programming on computer. Besides, a generalization of the method is also given by introducing a parameter.

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. We shall assume operator A to be bounded and self-adjoint, i.e. $A^* = A$, where A^* is the adjoint of A. We suppose, in addition, that operator A is positive, i.e.

$$(Au, u) \geqslant 0 \quad \text{for any } u \neq 0 \text{ of } H.$$

The problem is to find eigenvalues λ and corresponding eigenvectors x of H (provided that such λ and x exist) satisfying the following equation:

(1) $$Ax = \lambda x, \quad x \neq 0.$$

It follows from (1) that

$$\lambda(x) = \frac{(Ax, x)}{(x, x)}.$$

Hence, we get the following equation instead of (1).

$$Ax = \lambda(x)\, x, \quad x \neq 0$$

Instead of this equation we shall solve the following homogeneous equation

(2) $$\|Ax - \lambda(x)\, x\|^2 = 0 \quad x \neq 0$$

Thus, we see that if x is a solution of Eq. (2), then x is an eigenvector of operator A and conversely.

We shall now define an iterative method of solving Eq. (2). Let x_0 be an arbitrary vector of H such that $\|x_0\| = 1$.

Let us put

$$x^{n+1} = x_n - \frac{(y_n, y_n)}{(Ay_n, y_n)} y_n, \tag{3}$$

where

$$y_n = Ax_n - (Ax_n, x_n) x_n,$$

and where

$$x_{n+1} = \frac{x^{n+1}}{\|x^{n+1}\|}.$$

The sequence of approximate solutions x_n of Eq. (2) is then defined by relations (3).

Let us put

$$a_n = (Ax_n, x_n),$$

where x_n is as defined by (3).

We shall show that the sequence of numbers a_n is convergent. We have

$$a_{n+1} = (Ax_{n+1}, x_{n+1}) = \frac{(Ax^{(n+1)}, x^{(n+1)})}{(x^{(n+1)}, x^{(n+1)})}.$$

Thus, we get

$$a_{n+1} = \frac{a_n - 2t_n (y_n, Ax_n) + t_n^2 (Ay_n, y_n)}{1 + t_n^2 (y_n, y_n)}. \tag{4}$$

since

$$(y_n, x_n) = 0,$$

where

$$t_n = \frac{(y_n, y_n)}{(Ay_n, y_n)}. \tag{4'}$$

Let us observe that

$$(y_n, Ax_n) = (y_n, y_n).$$

Hence, in virtue of (4), it follows that

$$a_{n+1} = a_n - t_n \frac{t_n a_n (y_n, y_n) - t_n (Ay_n, y_n) + 2(y_n, y_n)}{1 + t_n^2 (y_n, y_n)}$$

Hence, we obtain

$$a_{n+1} = a_n - t_n \frac{(y_n, y_n)(1 + t_n a_n)}{1 + t_n^2 (y_n, y_n)}. \tag{5}$$

Since the numbers t_n and a_n are non-negative, the sequence (a_n) is decreasing. But it is also bounded and thus convergent.

It follows from (5) that the sequence of

$$t_n (y_n, y_n) \frac{1 + t_n a_n}{1 + t_n^2 (y_n, y_n)}$$

converges to zero.

Since

$$t_n \geqslant \frac{1}{\|A\|},$$

we get

$$(y_n, y_n) \to 0 \quad \text{as} \quad n \to \infty. \tag{6}$$

Denote by λ the limit of the sequence (y_n), i.e.

$$a_n \to \lambda \quad \text{as} \quad n \to \infty. \tag{7}$$

Since $\|x_n\| = 1$, one can choose a subsequence (x_{n_k}) convergent weakly to some element x^* of H. If $x \neq 0$, then it follows from relations (6) and (7) that x^* is an eigenvector corresponding to the eigenvalue λ. Suppose that (x_n) does not converge weakly to 0. It is easy to see that if λ is a simple eigenvalue of operator A, then the sequence of approximate eigenvectors x_n defined by process (3) converges weakly to the eigenvector x^* corresponding to λ. If the eigenvalue λ is not simple, then there exists a subsequence of the sequence (x_n) which converges weakly to x^*. Thus, we have proved the following

THEOREM 1. *If A is a self-adjoint linear bounded operator satisfying the following condition*

$$(Au, u) \geqslant 0 \quad \textit{for any } u \textit{ of } H \textit{ and } (Au, u) = 0 \textit{ implies } u = 0, \tag{8}$$

then the sequence of numbers (Ax_n, x_n) converges to a number λ and $Ax_n - (Ax_n, x_n)x_n \to 0$. If (x_n) does not converge weakly to 0, λ is an eigenvalue. The initial vector x_0 can be chosen arbitrarily. If λ is a simple eigenvalue of A, the sequence (x_n) converges weakly to the eigenvector x^* corresponding to the eigenvalue λ. If the eigenvalue λ is not simple, there exists a subsequence of (x_n) weakly convergent to x^*.

Remark. If condition (8) is not satisfied, we replace operator A by operator

$$B = A + aI, \tag{9}$$

where a is a properly chosen number, I being the identity mapping. If number a is so chosen that condition (8) is satisfied for B, we can apply Theorem 1 to operator B.

Suppose in addition that operator A is completely continuous. Suppose also that number a in (9) is so chosen that operator B satisfies condition (8). Then the following theorem holds

THEOREM 2. *Let x_0 be an arbitrary vector of H, and put*

$$x^{n+1} = x_n - \frac{(y_n', y_n')}{(By_n', y_n')} y_n',$$

where

$$y_n' = Bx_n - (Bx_n, x_n)\, x_n,$$

and where

$$x_{n+1} = \frac{x^{(n+1)}}{\|x^{(n+1)}\|}, \qquad n = 0, 1, 2, \ldots$$

Then the sequence of numbers (Bx_n, x_n) converges to an eigenvalue λ of operator B and the sequence (x_n) converges strongly to an eigenvector x^* corresponding to λ. If the eigenvalue λ is not simple, then x_0 is not supposed to be orthogonal to the subspace of the eigenvectors corresponding to λ.

Suppose now that operator A is defined by a real and symmetric matrix. Then applying the same argument as in Theorems 1 and 2 we get the following

THEOREM 3. *Suppose that the matrix A is real symmetric and positive definite. Then the sequence of numbers (Ax_n, x_n) converges to an eigenvalue λ of the matrix A and the sequence (x_n) defined by* (3) *converges to an eigenvector x^* corresponding, to λ. The initial vector x_0 can be chosen arbitrarily, but if the eigenvalue λ is not simple then the vector x_0 is not supposed to be orthogonal to the subspace of eigenvectors corresponding to λ.*

The general case of a real and symmetric matrix can be reduced to that considered in Theorem 3. One replaces then the matrix A by the matrix $B = A+aI$, where the number a is so chosen that B becomes positive definite. Then we can apply Theorem 3 to the matrix B.

We shall now introduce a parameter α in the process defined by formula (3). Thus, we get instead of (3)

$$x^{(n+1)} = x_n - \alpha t_n y_n, \tag{10}$$

where t_n is determined by formula (4′) and where y_n and x_n are the same as in (3). Simple calculation gives the following relation instead of (5)

$$a_{n+1} = a_n - \alpha t_n \frac{(y_n, y_n)(2 - \alpha(1 - a_n t_n))}{1+\alpha^2 t_n^2 (y_n, y_n)}.$$

Hence, we see that the sequence (a_n) is decreasing for positive α if

$$2 - \alpha(1 - a_n t_n) > 0. \tag{11}$$

Suppose now that operator A is self-adjoint and positive definite, i.e.

$$m(u, u) \leqslant (Au, u) \leqslant M(u, u),$$

where m and M are the minimum and the maximum eigenvalues of A, respectively Then we have, by (4′),

$$\frac{m}{M} \leqslant a_n t_n \leqslant \frac{M}{m}.$$

Hence, it follows that condition (11) is satisfied if

$$0 < \alpha < 2\left(1 - \frac{m}{M}\right)^{-1}. \tag{12}$$

Thus, we see that if condition (12) is satisfied, then the iterative method defined by formula (10) is still convergent and Theorems 1—3 can be stated for this process. Let us remark that in order to obtain Theorem 2 for the method defined by formula (10) we should replace m and M in condition (12) by the minimum and maximum eigenvalues of operator B defined by (9).

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MATHEMATICS

Two Iterative Methods for Linear Algebraic Equations

by

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Presented by W. ORLICZ on August 31, 1961

1. This paper contains two iterative methods of solving systems of linear algebraic equations. The first one concerns symmetric positive definite matrices. The coefficients of the corrections improving the approximate solutions of the system are obtained in the same way as in the method of steepest descent [1], i.e., by minimizing the error measure which is the same quadratic functional as in the method of steepest descent. However, the correction itself is here not the gradient of the quadratic functional at the corresponding point. Nevertheless, this correction is closely connected with the gradient and is called the sign-gradient, because its coordinates are just the signs of the gradient. By using the sign-gradient we obtain a simple method. The advantage of this method is that it requires very few multiplications. In fact, each iteration step requires only N multiplications, where N is the order of the matrix. This means that the number of multiplications is even less than that for the one step-method which requires $2N$ multiplications for each iteration step. Let us remark that the one step method changes actually only one coordinate at each iteration, while the sign-gradient method presented here changes all of them.

The second method concerns arbitrary non-singular matrices and uses also the notion of the sign-gradient. However, the coefficients mentioned above are obtained by minimizing the residuum at each iteration (see [2]). Besides, we consider also these methods with a numerical parameter.

Let A be a real symmetric and positive definite matrix

$$A = (a_{ik}), \qquad i, k = 1, 2, \ldots, N.$$

Denote by m, M the minimum and maximum eigenvalues of A, respectively. Since A is positive definite, we have

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x), \tag{1}$$

where $m > 0$.

Consider the linear equation

$$Ax = b, \tag{2}$$

where x and b are vectors in the N—dimensional Euclidean space E . Denote by x^* the solution of Eq. (2). Let us put

$$r = Ax - b \tag{3}$$

and consider the quadratic functional

$$F(x) = (r, A^{-1} r) = (Ax - b, x - x^*), \tag{4}$$

where A^{-1} is the inverse of A. $F(x)$ is to be considered as an error measure for the approximate solution of Eq. (2), and $2r$ is the gradient of the functional $F(x)$. Let x_n, y_n be arbitrary vectors in E_n and let α_n be an arbitrary real number. We get by a simple calculation in virtue of (4) that

$$F(x_n+\alpha_n y_n) = F(x_n) - 2\alpha_n (r_n, y_n)+\alpha_n^2 (Ay_n, y_n). \tag{5}$$

Let us put

$$y_n = (\operatorname{sign} r_{ni}), \qquad i = 1, 2, ..., N, \tag{5'}$$

where

$$r_n = (r_{ni}), \qquad i = 1, 2, ..., N,$$

r_{ni} being the coordinates of r_n. The vector y_n is called the sign-gradient. The sequence of the approximate solutions of Eq. (2) is then defined by the following iterative formula

$$x_{n+1} = x_n - \alpha_n y_n, \tag{6}$$

where α_n is so chosen as to minimize $F(x_{n+1})$. Hence, we get

$$\alpha_n = \frac{(r_n, y_n)}{(Ay_n, y_n)}.$$

Let us put

$$|x| = \sum_{i=0}^{N} |\xi_i|,$$

where

$$x = (\xi_i), \qquad i = 1, 2, ..., N.$$

Then we get, instead of (6),

$$x_{n+1} = x_n - \frac{|r_n|}{(Ay_n, y_n)} y_n. \tag{7}$$

It follows from (5) that

$$F(x_{n+1}) = Fx_n - \frac{|r_n|^2}{(Ay_n, y_n)}. \tag{8}$$

We have

$$\frac{F(x_{n+1})}{F(x_n)} = 1 - \frac{|r_n|^2}{(r_n, A^{-1} r_n)(Ay_n, y_n)} \tag{9}$$

and

$$\frac{|r_n|^2}{(r_n, A^{-1} r_n)(Ay_n, y_n)} \geqslant \frac{m |r_n|^2}{M \|r_n\|^2 N}. \tag{9'}$$

Let x_0 be an arbitrary initial vector. It follows from (9) that the sequence of the approximate solutions x_n defined in (7) converges to the solution x^* of Eq. (2). However, the ratio in (9′) depends on the ratio $|r_n|^2/\|r_n\|^2$ which varies between 1 and N, i.e.,

$$1 \leqslant \frac{|r_n|^2}{\|r_n\|^2} \leqslant N.$$

For the computations we use the following formula in addition to formula (7)

$$r_{n+1} = r_n - \frac{|r_n|}{(Ay_n, y_n)} Ay_n. \tag{10}$$

Remark. It follows from the foregoing that the computation scheme is given by formulae (7) and (10). Hence, we see that to carry out one iteration we do not need multiplications in the formula (7) and we have only N multiplications in the formula (10). Thus, each iteration for the sign-gradient method in question requires only N multiplications and one division. We do not take into consideration additions and subtractions.

We shall now introduce a numerical parameter a in the iterative method defined by formulae (7) and (10). Thus, we consider the iterative method defined by the following two formulae

$$x_{n+1} = x_n - a \frac{|r_n|}{(Ay_n, y_n)} y_n, \tag{11}$$

$$r_{n+1} = r_n - a \frac{|r_n|}{(Ay_n, y_n)} Ay_n. \tag{12}$$

where y_n denotes the sign-gradient defined in (5′). By a simple calculation we get, instead of the relation (8),

$$F(x_{n+1}) = F(x_n) - a(2 - a) \frac{|r_n|^2}{(Ay_n, y_n)}.$$

Hence, it follows that the iterative method defined by the formulae (11) and (12) is convergent for the values of the parameter a satisfying the following inequalities: $0 < a < 2$.

On can consider the case in the same way as above when the matrix A is Hermitian.

2. Consider now the case when A is an arbitrary non-singular matrix. We shall use the same general iteration formula (6). Then for arbitrary α_n and y_n we get the following relation

$$\|r_{n+1}\|^2 = \|r_n\|^2 - 2\alpha_n(r_n, Ay_n) + \alpha_n^2 \|Ay_n\|^2 \tag{13}$$

instead of (5). Choosing now the coefficient α_n so as to minimize $\|r_{n+1}\|^2$ we get

$$\alpha_n = \frac{(A^* r_n, y_n)}{\|Ay_n\|^2},$$

where A^* is the adjoint of A.

The gradient of the functional $\|r\|^2 = \|Ax - b\|^2$ is $2A^*r$, where $r = Ax - b$ is the residuum vector. As in par. 1 instead of the gradient A^*r we take the sign-gradient y_n consisting of the signs of the coordinates of A^*r, i.e.

$$y_n = \big(\operatorname{sign}(A^*r_n)_i\big), \qquad i = 1, 2, \ldots, N,$$

where

$$A^*r_n = \big((A^*r_n)_i\big), \qquad i = 1, 2, \ldots, N.$$

Thus, we obtain the following iteration formulae

$$x_{n+1} = x_n - \frac{|A^*r_n|}{\|Ay_n\|^2}\, y_n \tag{14}$$

and

$$r_{n+1} = r_n - \frac{|A^*r_n|}{\|Ay_n\|^2}\, Ay_n. \tag{15}$$

Hence, we get instead of the relation (13) the following one

$$\|r_{n+1}\|^2 = \|r_n\|^2 - \frac{|A^*r_n|^2}{\|Ay_n\|^2}. \tag{16}$$

Since the sequence of $\|r_n\|^2$ is decreasing and bounded, it has a limit. It follows from (16) that

$$|A^*r_n| \to 0 \quad \text{as} \quad n \to \infty.$$

Since A is non-singular, we get

$$r_n \to 0 \quad \text{as} \quad n \to \infty,$$

and, consequently, the method defined by the formulae (14) and (15) is convergent.

It is easy to see that

$$|A^*r_n| \geqslant \frac{|x_n - x^*|}{\|(A^*A)^{-1}\|}.$$

Hence it follows, in virtue of (16), that

$$\frac{\|r_{n+1}\|^2}{\|r_n\|^2} = 1 - \frac{|A^*r_n\|^2}{\|r_n\|^2\, \|Ay_n\|^2}$$

and we have

$$\frac{|A^*r_n|^2}{\|r_n\|^2\, \|Ay_n\|^2} \geqslant \frac{m^2\, |x_n - x^*|^2}{M^2\, \|x_n - x^*\|^2\, N},$$

where m and M are the minimum and maximum eigenvalues of the matrix A^*A, respectively.

As in par. 1, one can introduce a numerical parameter a in the method defined by formulae (14) and (15). Thus, we get

$$x_{n+1} = x_n - a\frac{|A^*r_n|}{\|Ay_n\|^2}\, y_n \tag{17}$$

and

$$r_{n+1} = r_n - a\frac{|A^*r_n|}{\|Ay_n\|^2}\, Ay_n. \tag{18}$$

By a simple calculation, we get instead of relation (16) the following one

$$\|r_{n+1}\|^2 = \|r_n\|^2 - a(2-a)\frac{|A^* r_n|^2}{\|Ay_n\|^2}.$$

Hence, it follows that the iterative method defined by the formulae (17) and (18) converges to the solution x^* of Eq. (2) if parameter a satisfies the following condition: $0 < a < 2$.

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REFERENCES

[1] L. V. Kantorovitch, *Functional analysis and applied mathematics* [in Russian]. Usp. Matem. Nauk (NS), 3 (1948), 89—185. [translated into English: National Bureau of Stand.].

[2] M. A. Krasnoselsky and S. G. Krein, *Iteration process with minimum residua* [in Russian]. Matem. Sbornik (NS), **31** (1952), 157—161.

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MATHEMATICS

A General Deflation Principle for Linear Iterations

by
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Iterative methods of solving large systems of linear algebraic equations play an important part in the modern matrix calculus. However, iteration methods have the great disadvantage that, very often, they converge too slowly and, therefore, the number of operations is large. Thus, there arises the problem of speeding-up their convergence. Various methods for this purpose are known. Some of them hold even for linear iterative methods in Hilbert space. The problem of speeding-up the convergence is also the concern of this paper. Although iterative methods of solving linear equations in Hilbert space are considered here, the main stress is laid on their application to matrix calculus. However, our approach to this problem is different. The main concept is based on introducing an auxiliary subspace in such a manner that the iteration process is carried out in some complementary subspace. In this way we obtain a deflation for the given iteration process. The auxiliary subspace can be introduced in many ways, and by a suitable choice one can improve the condition of the operator restricted to the subspace in which the process is acting. Thus, we get a speeding-up of the convergence of the iteration process. Three variants of the deflation principle are here considered. This principle can be easily applied to gradient methods of solving linear equations. Special cases of such application are contained in papers [1]—[5].

Let A be a linear (i.e. additive and homogeneous) operator with domain and range in a Hilbert space H. Consider the linear equation

$$Ax = b, \qquad x, b \in H. \tag{1}$$

We shall here consider the case when A is self-adjoint and positive definite, or, more precisely,

$$A^* = A,$$

where A^* is the adjoint of A, and

$$m(x, x) \leqslant (Ax, x) \leqslant M(x, x),$$

where $0 < m < M < \infty$, m and M being the bounds of A.

Without loss of generality one can suppose that $\|b\| = 1$.

1. The first variant of the deflation principle

Let L be an arbitrary subspace of H orthogonal to b and to Ab, i.e.

$$L \perp b \quad \text{and} \quad L \perp Ab.$$

Let us put

$$Ł = A(L)$$

i.e. $Ł$ is the image of the subspace L. Hence, it follows that the subspace $Ł$ is orthogonal to b.

Let Z be the complementary subspace of H orthogonal to b and to $Ł$, i.e.

$$Z \perp b \quad \text{and} \quad Z \perp Ł.$$

Denote by $P_b x$ the orthogonal projection of x of H onto the subspace spanned by the vector b. Denote by $P_Ł x$, $P_Z x$ the orthogonal projection of x onto the subspaces $Ł$, Z of H, respectively. We have

$$P_Z x = x - P_b x - P_Ł x. \tag{2}$$

Let us now define the following linear transformation R:

$$z = Rx = P_Z Ax = Ax - P_b Ax - P_Ł Ax.$$

Hence, we get the following expression for Rx

$$z = Rx = Ax - (Ax, b)\, b - A\tilde{x}, \tag{3}$$

where

$$A\tilde{x} = P_Ł Ax.$$

Consider now the following homogeneous equation

$$Rx = 0, \quad x \neq 0, \tag{4}$$

where operator R is as defined by (3).

We shall show that the solution of Eq. (1) can be reduced to that of Eq. (4).

Let x_0 be an arbitrary element of H satisfying the following condition.

$$(x_0, b) \neq 0. \tag{5}$$

Suppose now that the following iterative process for the approximate solution x_n of Eq. (4) is given:

$$x_{n+1} = x_n - a_n z_n, \quad n = 0, 1, 2, \ldots, \tag{6}$$

where

$$z_n = Rx_n.$$

Thus, we suppose that the sequence of approximate solutions x_n converges to a solution of Eq. (4), i.e.

$$x_n \to x \text{ as } n \to \infty \text{ and } Rx = 0. \tag{7}$$

It is easy to see that condition (5) implies the following one

$$\|x_n\| \geqslant |(x_0, b)| > 0. \tag{8}$$

In fact, we have, by (6),

$$x_n = x_0 - a_0 z_0 - \ldots - a_{n-1} z_{n-1}, \tag{9}$$

where $z_i \in Z$ for $i = 0, 1, 2, \ldots$ Since Z is orthogonal to b, we get

$$(x_n, b) = (x_0, b) \quad \text{for} \quad n = 0, 1, \ldots,$$

whence we get relation (8).

Suppose now that for x_n, defined by the formula in (6), the norm of the corresponding z_n is sufficiently small. This statement is true for sufficiently large n, since the sequence of z_n converges to the zero element. Then we calculate the approximate solution y_n of Eq. (1) by the following formula

$$y_n = \frac{1}{(Ax_n, b)} (x_n - \tilde{x}_n), \tag{10}$$

where x_n is defined by the relation

$$P_L Ax_n = A\tilde{x}_n. \tag{11}$$

It is easy to see that

$$(Ax_n, b) \to (Ax, b) \neq 0, \text{ as } n \to \infty,$$

where x is as defined by (7). In fact, we have, by (3),

$$z_n = Rx_n = A(x_n - \tilde{x}_n) - (Ax_n, b)\, b, \tag{12}$$

where $\tilde{x}_n$ is as defined by the relation in (11). It follows from (11) that

$$\tilde{x}_n \to \tilde{x} \quad \text{as} \quad n \to \infty,$$

where $P_L Ax = A\tilde{x}$, x being the solution of Eq. (4) obtained by relation (7). Since $A\tilde{x}$ is in L, or, equivalently, $\tilde{x}$ is in L, we get, by (12), the following equation

$$Rx = A(x - \tilde{x}) - (Ax, b)\, b = 0, \tag{13}$$

which is equivalent to Eq. (4). It follows from (9) that

$$x = x_0 - z, \quad z \in Z.$$

Thus, assuming that $(Ax, b) = 0$, we get, by (13),

$$x = x_0 - z = \tilde{x}, \tag{14}$$

where $\tilde{x} \in L$ and $z \in Z$. Hence, we get the following relation

$$(x_0, b) = 0,$$

which is a contradiction to (5).

Let x^* be the solution of Eq. (1). Then the error estimate for the approximate solution y_n of Eq. (1) determined by (10) is given by the following formula

$$\|Ay_n - b\| \leqslant \frac{\|z_n\|}{(Ax_n, b)}, \tag{15}$$

or

$$\|y_n - x^*\| \leqslant \frac{\|z_n\|}{m\,(Ax_n, b)}. \tag{15'}$$

It is easy to see that

$$(Rz, z) = (Az, z) \tag{16}$$

for any z of Z. Hence, we infer that operator R with domain and range in Z is self-adjoint and positive definite. Denote by m_Z, M_Z the bounds of R in Z, respectively. Thus, we get, by (16),

$$m \leqslant m_Z, \qquad M_Z \leqslant M$$

and for any z of Z we have

$$m_Z(z, z) \leqslant (Rz, z) \leqslant M_Z(z, z). \tag{17}$$

Thus, we see that transformation R defined by (3) gives the iterative process with deflation defined by the formula in (6). Relations (16) and (17) show that by a suitable choice of the auxiliary subspace L the condition number of operator R can be much better than that of operator A. Since in many iterative methods the speed of the convergence depends on the condition number of the operator, we get in this way a speeding-up of the convergence.

It follows from the foregoing that the deflation principle defined above gives a method of embedding the iteration process in a subspace Z. This embedding is accomplished by the transformation R. The subspace Z is called the iteration carrier. The deficiency of the subspace Z is called the index of the deflation. The subspace L is the auxiliary subspace. If the subspace L consists only of the zero element, then the index of the deflation is one.

We obtain various kinds of iterative methods with deflation by specializing the coefficients in the formula (6). Thus, it is easy to see that the method presented in [1] is a minimum residua method with deflation with index one. Analogously, the method of steepest orthogonal descent contained in [2] is a method of steepest descent with deflation with index one. Similarly, the general method with minimum ortho-residua given in [4] is a minimum residua method with deflation. It is easy to see that also the general method of steepest ortho-descent described in [3] is a steepest descent method with deflation.

Remark. Suppose that operator A is defined by the matrix

$$A = (a_{ik}), \qquad i, k = 1, 2, \ldots, N.$$

Let $l_1, l_2, \ldots, l_s$ be linearly independent vectors orthogonal to b and to Ab. Denote by $u_1, u_2, \ldots, u_s$ $(s < N)$ the orthonormal system obtained from $Al_1, Al_2, \ldots, Al_s$ by an orthogonalization process. The auxiliary subspace L is then the subspace spanned by the vectors l_i, and the orthonormal system forms a basis in the subspace L orthogonal to b. Then the operator R can be written in the following matrix form:

Put

$$b = (b_1, b_2, \ldots, b_N), \qquad c_k = (a_k, b), \qquad a_k = (a_{1k}, a_{2k}, \ldots, a_{Nk}),$$

$$u_j = (u_{1j}, u_{2j}, \ldots, u_{Nj}), \qquad d_{kl} = (a_k, u_j), \qquad j = 1, 2, \ldots, s.$$

Then we get

$$R = (r_{ik}), \qquad i, k = 1, 2, \ldots, N,$$

where

$$r_{ik} = a_{ik} - b_i c_k - l_{i1} d_{k1} - l_{i2} d_{k2} - \ldots - l_{is} d_{ks}.$$

2. The second variant of the deflation principle

Let L be an arbitrary subspace of H orthogonal to b, and put as in par. 1

$$\mathcal{L} = A(L), \quad \mathcal{L} \perp b.$$

L is then the auxiliary subspace. Let us put

$$b' = b - P_{\mathcal{L}}\, b, \tag{18}$$

where $P_{\mathcal{L}}$ is the orthogonal projection onto $\mathcal{L}$. If

$$P_{\mathcal{L}}\, b = A\tilde{b},$$

then we have,

$$b' = b - A\tilde{b}. \tag{19}$$

Let Z be the complementary subspace orthogonal to b' and to $\mathcal{L}$. Instead of Eq. (1) we shall consider the following equation

$$Ax = b'. \tag{20}$$

Thus, if x^* is the solution of Eq. (1) and x' is the solution of Eq. (20), then we have, by (19),

$$x^* = x' + \tilde{b}. \tag{21}$$

One can now apply to Eq. (20) the same argument as in par. 1. Denote by $P_{b'}\, x$ the orthogonal projection of x onto the one dimensional subspace containing b', it being defined in (19). Let Z be the complementary subspace of H orthogonal to b' and to $\mathcal{L}$. We get, instead of (3),

$$z = Rx = Ax - \frac{(Ax, b')}{(b', b')}\, b' - A\tilde{x}, \tag{22}$$

where

$$A\tilde{x} = P_{\mathcal{L}}\, Ax.$$

As in par. 1, choosing x_0 so as to satisfy condition (5) we define the sequence of approximate solutions x_n of Eq. (4) by the formula (6), and we assume its convergence to a solution of Eq. (4) in which R is defined by (22). Then the approximate solution of Eq. (22) is given by the following formula

$$y'_n = \frac{(b', b')}{(Ax_n, b')}(x_n - \tilde{x}_n), \tag{23}$$

where

$$P_{\mathcal{L}}\, Ax_n = A\tilde{x}_n.$$

Hence, it follows that the approximate solution y_n of Eq. (1) is given by the following relation:

$$y_n = y'_n + \tilde{b},$$

where y'_n, $\tilde{b}$ are defined in (23), (19), respectively.

Arguing as in par. 1, one can prove that

$$(Ax_n, b') \to (Ax, b') \neq 0, \text{ as } n \to \infty, \tag{24}$$

using (14) and the fact that $Z, L \perp b$. The error estimate for x' is given by (15) or (15′) in which b is replaced by $b'/(b', b')$.

The remark of par. 1 remains also true here, where the vectors l_i $(i \leqslant s)$ are supposed to be orthogonal only to b.

Paper [5] contains a steepest descent method with the second variant of deflation.

3. The third variant of the deflation principle

Let L be an arbitrary subspace of H. In the same way as in par. 2 we construct the subspace Z orthogonal to b', defined by (19), and to $L = A(L)$. Let us assume that the following condition is satisfied.

The subspace spanned by the elements of L and of Z is not the whole of H. In other words, there exists an element v of H such that

$$v \perp L \quad \text{and} \quad v \perp Z. \tag{25}$$

Condition (24) is satisfied if L is a finite-dimensional subspace of H. This case is very important for the application to the solution of a system of linear algebraic equations.

Arguing as in par. 2, let us start with an initial element x_0 satisfying the following condition instead of (5).

$$(x_0, v) \neq 0. \tag{26}$$

All assertions of par. 2 remain true also in this case, and the sequence of the approximate solutions x_n defined by (6) converges to a non-trivial solution of Eq. (22). In the same way as in par. 2 it follows from (25) and (26) that relation (24) is also true in this case. The remark of par. 2 concerning the matrix form of R remains also true here.

Paper [5] contains a steepest descent method with the third variant of deflation.

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(INSTYTUT MATEMATYCZNY, PAN)

REFERENCES

[1] M. Altman, *A linear iterative method with a vector parameter*, Bull. Acad. Polon. Sci., Sér. math., astr. et phys. **9** (1961), 169—174.

[2] — , *A method of steepest ortho-descent*, ibid., **9** (1961), 575—580.

[3] — , *A general method of steepest ortho-descent*, ibid., 645—652.

[4] — , *A general method with minimum ortho-residua*, ibid., 739—744.

[5] — , *A generalized method of steepest descent*, ibid., 733—738.

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MATHEMATICS

A Newton Type Method for Linear Equations

by

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Presented by W. ORLICZ on August 31, 1961

This paper contains an iterative method for solving linear bounded operator equations in Hilbert space. The formalism of this method is based on a generalization of Newton's classical method given in [1] and [2]. The generalization concerns the problem of finding zero elements of non-linear functionals on a Banach space. However, the examination of the method presented here requires an independent argument and the result obtained in this way does not follow from the general theory of our generalization of Newton's method. Besides, the method in question is actually also based on another idea of reducing the operator equation to a homogeneous one. This concept has first been used in [3]. The case considered there concerns linear equations with self-adjoint positive definite operators and the method itself is different. An application to the matrix calculus is also given. In the case of a system of linear algebraic equations the method in question is suitable for programming on computer.

Let A be a non-singular linear bounded operator with domain and range in a Hilbert space H. Consider the linear equation

$$Ax = b, \qquad x, b \in H. \tag{1}$$

Without loss of generality one can suppose that $\|b\| = 1$. Instead of Eq. (1) we shall consider the following equation

$$Ax = (Ax, b)\, b, \qquad x \neq 0, \qquad \|b\| = 1. \tag{2}$$

Thus, if x is a solution of Eq. (2), then $x/(Ax, b)$ is the solution of Eq. (1). It follows from (2) that $(Ax, b) \neq 0$, since operator A is non-singular.

We shall now define the iterative process of solving Eq. (2).

Denote by R the linear operator defined as follows:

$$Rx = Ax - (Ax, b)\, b, \qquad x \in H, \qquad \|b\| = 1. \tag{3}$$

Put

$$F(x) = \|Rx\|^2.$$

Then we get the following equation instead of Eq. (2)

$$F(x) = \|Rx\|^2 = \|Ax - (Ax, b)\, b\|^2 = \|Ax\|^2 - (Ax, b)^2 = 0. \tag{4}$$

Thus, the problem of solving Eq. (2) reduces to that of finding a non-trivial solution of the functional equation (4). An abstract generalization of Newton's method of solving such equations is given in [1]. We shall now apply this method. Let x_0 be an arbitrary element of H such that

$$(x_0, x^*) \neq 0, \tag{5}$$

where x^* is the solution of Eq. (1).

Then the sequence of approximate solutions x_n of Eq. (4) is defined as follows:

$$x_{n+1} = x_n - \frac{F(x_n)}{\|F'(x_n)\|^2} F'(x_n), \quad n = 0, 1, 2, \ldots, \tag{6}$$

where $F'(x)$ denotes the Fréchet derivative of $F(x)$. We have

$$y = \tfrac{1}{2} F'(x) = R^* Rx. \tag{7}$$

Thus we get, instead of (6),

$$x_{n+1} = x_n - \frac{F(x_n)}{2\|y_n\|^2} y_n, \tag{8}$$

where y_n is defined by the formula (7) in which x should be replaced by x_n.

It follows from (7) and (4) that

$$(y, x) = F(x). \tag{9}$$

Hence we get, instead of (8),

$$x_{n+1} = x_n - \frac{(y_n, x_n)}{2\|y_n\|^2} y_n, \quad y_n = R^* Rx_n. \tag{10}$$

In virtue of (10) and (7) we obtain

$$y_{n+1} = y_n - \frac{(y_n, x_n)}{2\|y_n\|^2} R^* Ry_n \tag{11}$$

Or, equivalently, since Rx is orthogonal to b for any x of H:

$$y_{n+1} = y_n - \frac{(y_n, x_n)}{2\|y_n\|^2} A^* Ry_n. \tag{11'}$$

where operator R is as defined by (3) and where R^* is the adjoint of R. We have then

$$R^* x = A^* x - (b, x) A^* b.$$

It follows from (10) that

$$\|x_{n+1}\|^2 = \|x_n\|^2 - \tfrac{3}{4} \frac{(y_n, x_n)^2}{\|y_n\|^2}. \tag{12}$$

Thus it follows that the sequence of $\|x_n\|^2$ is decreasing and bounded and, consequently, convergent. Relation (12) implies

$$\|x_{n+1}\|^2 = \|x_0\|^2 - \tfrac{3}{4} \sum_{i=0}^{n} \frac{(y_i, x_i)^2}{(y_i, y_i)}, \tag{13}$$

Therefrom follows the convergence of the series

$$\sum_{i=0}^{\infty} \frac{(y_i, x_i)^2}{(y_i, y_i)} < \infty. \tag{14}$$

In virtue of (13) we infer that

$$x_n \to 0 \quad \text{as} \quad n \to \infty$$

if and only if

$$\|x_0\|^2 = \tfrac{3}{4} \sum_{i=0}^{\infty} \frac{(y_i, x_i)^2}{(y_i, y_i)}.$$

Since

$$y_n = R^* Rx_n = A^* Rx_n,$$

we get

$$(y_n, x^*) = (A^* Rx_n, x^*) = (Rx_n, b) = 0 \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{15}$$

We have, by (10),

$$x_n = x_0 - a_0 y_0 - \ldots - a_{n-1} y_{n-1}, \quad a_i = \frac{(y_i, x_i)}{2(y_i, y_i)}. \tag{16}$$

Hence, we get, by (5), (15) and (16),

$$(x_n, x^*) = (x_0, x^*) \neq 0$$

and, consequently, there is a number a such that

$$\|x_n\| \geqslant a > 0 \quad \text{for} \quad n = 0, 1, 2, \ldots \tag{17}$$

It follows from (17) and (13) that condition (5) implies

$$\tfrac{3}{4} \sum_{i=0}^{\infty} \frac{(y_i, x_i)^2}{(y_i, y_i)} < \|x_0\|^2.$$

From (12) it follows in virtue of (9) that

$$F(x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

We have, by (14),

$$(F(x_n))^{1/2} = |(Ax_n, b)| \, \|A\tilde{y}_n - b\| \to 0 \quad \text{as} \quad n \to \infty,$$

where

$$\tilde{y}_n = \frac{x_n}{(Ax_n, b)} = \frac{x_n}{(x_n, A^* b)}. \tag{18}$$

Thus, we see that $\tilde{y}_n$ defined in (18) is the approximate solution of Eq. (1). The residuum for x_n is given by the following expression

$$\frac{(F(x_n))^{1/2}}{|(x_n, A^* b)|} = \frac{(y_n, x_n)^{1/2}}{|(x_n, A^* b)|}. \tag{19}$$

Since operator A is non-singular, it follows that the numbers $(x_n, A^* b)$ do not vanish for sufficiently large n.

Operator $T = R^* R$ is non-singular on the subspace orthogonal to the solution of Eq. (1) and it maps this subspace onto iself. Let us put $y_n = Tz_n$, where z_n is orthogonal to the solution of Eq. (1). Then in relation (11) y_n can be replaced by z_n and we can apply the same argument as in [3].

Remark. Suppose that operator A is defined by the matrix

$$A = (a_{ik}), \quad i, k = 1, 2, ..., N.$$

Then operator R can be written in the following matrix form

$$R = (r_{ik}), \quad i, k = 1, 2, ..., N,$$

where

$$r_{ik} = a_{ik} - b_i c_k,$$

$$b = (b_1, b_2, ..., b_N), \quad c_k = (a_k, b),$$

$$a_k = (a_{1k}, a_{2k}, ..., a_{Nk}), \quad k = 1, 2, ..., N.$$

Let us note the following order in which the computations should be carried out on computer.

1. Compute and store vector $A^* b$.

2. Compute and store numbers c_k ($k = 1, 2, ..., N$) until matrix R is completely calculated.

3. Compute and store matrix R replacing A by R.

4. Now, carry out iterations using the formula (10). Having finished the iterations use formula (18); formula (19) gives the error measure.

Let us remark that the known gradient methods, like the method of steepest descent, the method with minimum residua and other methods use actually two iteration formulas: one of them concerns the approximate solution of the equation and corresponds to formula (10); the second iteration formula concerns the gradient and corresponds to formula (11).

However, in the method described above the iteration formula (11) is not needed at all for the computations because everything is contained in formula (10). Thus, the advantage of the method presented here is that its program on computer is much shorter.

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REFERENCES

[1] M. Altman, *Concerning approximate solutions of non-linear functional equations*, Bull. Acad. Sci., Cl. III, **5** (1957), 461—465.

[2] — , *On the approximate solutions of non-linear functional equations in Banach spaces*, ibid., Sér. sci. math., astr. et phys., **6** (1958), 19—24.

[3] — , *A linear iterative method with a vector parameter*, ibid., **9** (1961), 267—271.

БЮЛЛЕТЕНЬ
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МАТЕМАТИКА

М. БУРНАТ

УСТОЙЧИВОСТЬ СОБСТВЕННЫХ ФУНКЦИЙ И СПЕКТР ДЛЯ ОПЕРАТОРА ХИЛЛА

Представлено Т. ВАЖЕВСКИМ 2 сентября 1961 г.

В квантовой теории кристаллов рассматривается следующая задача на собственные значения (уравнение Хилла):

$$-u''+q(x)u=\lambda u \qquad (-\infty<x<+\infty), \tag{1}$$

где $q(x)$ — периодическая и непрерывная функция.

Решение $u(x)$ уравнения (1), определенное для $-\infty<x<+\infty$, называется устойчивым, если $u(x)$ — ограниченная функция. В противоположном случае решение уравнения (1) называется неустойчивым. Вещественное число λ называется собственным значением, а соответствующее решение уравнения (1) — собственной функцией, если это решение является устойчивым.

Множество всех собственных значений будем обозначать Λ_c, множество всех остальных вещественных чисел — Λ_p. В квантовой теории кристаллов множество Λ_c играет важную роль. Винтнер в своей работе [1] рассматривает самосопряженное в $L^2(-\infty, +\infty)$ расширение Af оператора $-u''+q(x)u$ при вышеуказанных предположениях относительно $q(x)$. Пусть $S(A)$ — спектр оператора Af. Как известно (см. [2] Кодаира), спектр $S(A)$ состоит исключительно из полос непрерывного спектра.

В работе [1] сообщается без доказательства, что $\Lambda_c=S(A)$, причем автор пишет, что доказательство довольно длинно и затруднительно. Целью настоящей заметки является простое доказательство, что $\Lambda_c=S(A)$.

Докажем следующие леммы:

ЛЕММА 1. *Если $\lambda\in\Lambda_c$, тогда для всякого нетривиального решения уравнения* (1) *имеет место*

$$\int_0^{+\infty}|u(x)|^2\,dx=+\infty. \tag{2}$$

ЛЕММА 2. *Если $\lambda\in\Lambda_p$, тогда существует система линейно независимых решений уравнения* (1) *вида*

$$u_1(x)=e^{kx}v_1(x), \qquad u_2(x)=e^{-kx}v_2(x),$$

где k — положительное вещественное число, а $v_1(x)$, $v_2(x)$ — периодические функции.

Доказательство лемм

Пусть $\varphi_1(x)$ и $\varphi_2(x)$ линейно независимая система решений уравнения (1). Тогда вронскиан

$$W(\varphi_1, \varphi_2) = \begin{vmatrix} \varphi_1, & \varphi_2 \\ \varphi_1', & \varphi_2' \end{vmatrix} \equiv \text{const} \neq 0.$$

Поскольку $q(x)$ — периодическая функция с периодом a, то $\varphi_1(x+a)$ и $\varphi_2(x+a)$ также являются решениями уравнения (1), и

$$\varphi_i(x+a) = \alpha_{i,1} \cdot \varphi_1(x) + \alpha_{i,2} \cdot \varphi_2(x) \qquad (i = 1, 2),$$

где α_{ik} — постоянные числа. Далее имеем

$$W(\varphi_1(x+a)\varphi_2(x+a)) = W(\varphi_1(x)\varphi_2(x)) \cdot \det(\alpha_{ik}),$$

откуда получаем, что $\det(\alpha_{ik}) = 1$. Таким образом, уравнение (1) допускает тогда и только тогда решение $\Psi(x)$, удовлетворяющее условию $\Psi(x+a) = \mu\Psi(x)$ (μ — постоянная), когда

$$\mu^2 - (\alpha_{11} + \alpha_{22})\mu + 1 = 0. \tag{3}$$

Возможны три случая:

1. $\alpha_{11} + \alpha_{22} > 2$. Тогда существуют два линейно независимых решения уравнения (1), которые имеют вид:

$$\Psi_1(x) = e^{kx} v_1(x), \qquad \Psi_2(x) = e^{-kx} v_2(x),$$

где k — вещественное число, заданное равенством $\mu_j = ka \cdot \exp(-1)^{j+1}$ ($j = 1, 2$) (μ_j решения уравнения (3)), $v_1(x)$ и $v_2(x)$ — периодические функции периода a.

2. $\alpha_{11} + \alpha_{22} < 2$. В этом случае существуют два линейно независимые решения, имеющие вид:

$$\Psi_1(x) = e^{ikx} v_1(x), \qquad \Psi_2(x) = e^{-ikx} v_2(x),$$

где k — вещественное число, заданное равенством $\mu_j = ka \cdot \exp i(-1)^{j+1}$ ($j = 1, 2$) (μ_j решения уравнения (3)), $v_1(x)$ и $v_2(x)$ — периодические функции периода a.

3. $\alpha_{11} + \alpha_{22} = 2$. Тогда существует, по крайней мере, одно периодическое решение $\Psi(x)$ уравнения (1). Пусть $\Psi^*(x)$ — решение, для которого $W(\Psi, \Psi^*) \neq 0$. Тогда

$$\Psi^*(x+a) = \alpha\Psi^*(a) + \beta\Psi(a)$$

$$\Psi(x+a) = \Psi(x),$$

где $\begin{vmatrix} \alpha, & \beta \\ 0, & 1 \end{vmatrix} = \alpha = 1$. Таким образом

$$\Psi^*(x+a) - \Psi^*(x) = \beta\Psi(x).$$

Если $\beta = 0$, тогда $\Psi^*(x)$ — периодическая функция. В противоположном случае, на основании равенства

$$\Psi^*(x+na) - \Psi^*(x) = \sum_{l=1}^{n} \Psi^*(x+la) - \Psi^*(x+(l-1)a) = n\beta\Psi(x),$$

получаем, что $\Psi^*(x)$ — неограниченная и удовлетворяющая условию (2) функция.

Из определения множеств Λ_c и Λ_p следуют непосредственно Леммы 1 и 2.

Докажем, что

(4) $$\Lambda_c \subset S(A).$$

Действительно, если $\lambda \notin S(A)$, тогда существует оператор

$$(A-\lambda I)^{-1} f = \int_{-\infty}^{+\infty} H(x, s, \lambda)\, f(s)\, ds,$$

где для каждого $s \in (-\infty, +\infty)$ функция $H(x, s, \lambda)$ дважды непрерывно дифференцируема по $x \in (-\infty, s)$ и $x \in (s, +\infty)$, причем имеется для $x \neq s$ равенство

$$-\frac{d^2(H(x, s, \lambda))}{dx^2} + q(x)\, H(x, s, \lambda) = \lambda H(x, s, \lambda).$$

Кроме того, функция $H(x, s, \lambda)$ является, как известно, ядром типа Карлемана и, следовательно, имеем

$$\int_{-\infty}^{+\infty} |H(x, s, \lambda)|^2 \quad dx < +\infty.$$

Таким образом, поскольку для некоторого s и $x \geqslant 0$ $H \neq 0$, тогда, на основании Леммы 1, имеется $\lambda \notin \Lambda_c$.

Далее имеет место

(5) $$S(A) \subset \Lambda_c.$$

Действительно, пусть $\lambda \notin \Lambda_c$, то есть $\lambda \in \Lambda_p$. Тогда, на основании Леммы 2, можно построить функцию

$$G(x, s, \lambda) = e^{-k|x-s|} \quad \Phi(x, s, \lambda),$$

где

$$\Phi(x, s, \lambda) = \begin{cases} v_1(x) \cdot v_2(s) & x < s, \\ v_1(s) \cdot v_2(x) & x > s. \end{cases}$$

Функции v_1, v_2 подобраны таким образом, чтобы $G(x, s, \lambda)$ имело для $x = s$ разрыв, характерный для функции Грина оператора $-u + q(x)u$.

Легко проверить, что интегральный оператор

$$Gf = \int_{-\infty}^{+\infty} G(x, s, \lambda)\, f(s)\, ds$$

является обратным для оператора $A - \lambda I$. Поскольку оператор Gf является ограниченным в $L^2(-\infty, +\infty)$ оператором, получаем, что существует оператор $(A - \lambda I)^{-1}$, который является ограниченным. Значит $\lambda \notin S(A)$.

КАФЕДРА МАТЕМАТИЧЕСКОГО АНАЛИЗА, ВАРШАВСКИЙ УНИВЕРСИТЕТ
(KATEDRA ANALIZY MATEMATYCZNEJ, UNIWERSYTET WARSZAWSKI)

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

[1] D. Wintner, *Stability and spectrum in the wave mechanics of lattices*, Phys. Rev., **72** (1947), 81—82.

[2] Kunihiko Kodaira, *The eigenvalue problem for ordinary differential equations of the second order and Heisenbergs theory of S-matrices*, Amer. J. of Math., **71** (1949), 921—945.

M. BURNAT. THE STABILITY OF EIGENFUNCTIONS AND THE SPECTRUM OF HILL'S EQUATION

Let us consider the eigenproblem of the form

$$-u'' + q(x)u = \lambda u, \qquad -\infty < x < +\infty, \tag{1}$$

where $q(x)$ denotes a periodic and continuous function (Hill's equation). In the quantum theory of crystals we consider a set of real numbers Λ_c consisting of numbers λ such that Eq. (1) has for them a limited solution, different from 0.

We denote by S the spectrum of selfadjoint extension in $L^2(-\infty, +\infty)$ of the operator $-u'' + q(x)u$.

The equality $S = \Lambda_c$ is proved.

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MATHEMATICS

A Note on Some Relations between Fourier and Hankel Transforms

by

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Presented by K. KURATOWSKI on September 13, 1961

1. Introduction

In this paper we are concerned with the relations existing among the Fourier cosine transform, the Fourier sine transform and the Hankel transform of a function. These transforms are defined, respectively, by the equations

$$F_c\{f(x);\xi\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos(\xi x)\,dx, \tag{1}$$

$$F_s\{f(x);\xi\} = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\xi x)\,dx, \tag{2}$$

$$H_\nu\{f(x);\xi\} = \int_0^\infty xf(x)\,J_\nu(\xi x)\,dx. \tag{3}$$

We shall also make use of the exponential (or complex) Fourier transform defined by the equation

$$F\{f(x);\xi\} = \sqrt{\frac{1}{2\pi}}\int_{-\infty}^\infty f(x)\,e^{i\xi x}\,dx. \tag{4}$$

From these transforms we can construct multiple transforms in an obvious way. For instance, we can construct the double transform

$$P_c\{f(x,z);\xi,\zeta\} = F[F_c\{f(x,z);\xi\};\zeta] = \frac{1}{\pi}\int_{-\infty}^\infty e^{i\xi z}\,dz\int_0^\infty f(x,z)\cos(\xi x)\,dx. \tag{5}$$

In a similar way we can define the double transforms

$$P_s\{f(x,z);\xi,\zeta\} = F[F_s\{f(x,z);\xi\};\zeta], \tag{6}$$

$$A_\nu\{f(x,z);\xi,\zeta\} = F[H_\nu\{f(x,z);\xi\};\zeta]. \tag{7}$$

The P and A notation is suggested by the fact that transforms of the former type arise in certain plane problems of mathematical physics, while those of the latter type arise in problems in which there is axial symmetry.

Similarly, in certain dynamical problems we encounter triple transforms of the types

$$D_c\{f(x,z,t);\xi,\zeta,\omega\} = F[P_c\{f(x,z,t);\xi,\zeta\};\omega], \tag{8}$$

$$D_s\{f(x,z,t);\xi,\zeta,\omega\} = F[P_s\{f(x,z,t);\xi,\zeta\};\omega], \tag{9}$$

$$D_\nu\{f(x,z,t);\xi,\zeta,\omega\} = F[A_\nu\{f(x,z,t);\xi,\zeta\};\omega]. \tag{10}$$

We shall also use the notation of the inverse transform, i.e. if, for example, $A_\nu\{f(x,z);\xi,\zeta\} = g(\xi,\zeta)$, we shall write $f(x,z) = A_\nu^{-1}\{g(\xi,\zeta);x,z\}$.

Throughout we shall assume that the functions being considered belong to the class of functions for which the relevant transforms exist, i.e. if we are considering the relation between a Fourier cosine transform and a Hankel transform of zero order, we are tacitly assuming that we are restricting our considerations to those functions for which the relevant Fourier and Hankel integrals exist.

2. Relations between Fourier and Hankel transforms

From the theory of Bessel functions ([1], p. 405), we know that

$$\int_0^\infty J_0(\varrho\xi)\cos(x\xi)\,d\xi = \begin{cases} (\varrho^2 - x^2)^{-\frac{1}{2}}, & \text{if } \varrho > x, \\ 0, & \text{if } \varrho < x, \end{cases} \tag{11}$$

where it is assumed that both ϱ and x are positive.

If we invert equation (11) by means of the Hankel inversion theorem, ([2], p. 48), we find that

$$\int_0^\infty \frac{\varrho J_0(\varrho\xi)\,d\varrho}{\sqrt{(\varrho^2 - x^2)}} = \xi^{-1}\cos(x\xi).$$

Multiplying both sides of this relation by $(2/\pi)^{1/2} f(x)$, and integrating with respect to x from 0 to ∞, we find that

$$\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\,dx \int_0^\infty \frac{\varrho J_0(\varrho\xi)\,d\varrho}{\sqrt{(\varrho^2 - x^2)}} = \xi^{-1} F\{f(x);\xi\}.$$

If we interchange the order of integrations in the integral on the left-hand side of this equation, we find that

$$H_0\{g(\varrho);\xi\} = \xi^{-1} F_c\{f(x);\xi\},$$

where

$$g(\varrho) = \sqrt{\frac{2}{\pi}}\int_0^\varrho \frac{f(x)\,dx}{\sqrt{(\varrho^2 - x^2)}}.$$

If we introduce the transformation *)

$$T_1\{\varphi(x, ...); \varrho\} = \sqrt{\frac{2}{\pi}} \int_0^{\varrho} \frac{\varphi(x, ...)\, dx}{\sqrt{(\varrho^2 - x^2)}} \tag{12}$$

we see that we can write this result in the form

$$H_0\{g(\varrho); \xi\} = \xi^{-1} F_c\{f(x); \xi\}, \quad \text{if} \quad g(\varrho) = T_1\{f(x); \varrho\}. \tag{13}$$

On the other hand, if we invert equation (11) by means of the Fourier cosine inversion theorem ([2], p. 17), we find that

$$\frac{2}{\pi} \int_0^{\infty} \frac{\cos(\xi x)\, dx}{\sqrt{(\varrho^2 - x^2)}} = J_0(\varrho\xi).$$

Multiplying both sides of this equation by $\varrho f(\varrho)$ and integrating with respect to ϱ from 0 to ∞, and introducing the transformation T_2 defined by the equation

$$T_2\{\psi(\varrho, ...); x\} = \sqrt{\frac{2}{\pi}} \int_x^{\infty} \frac{\varrho\psi(\varrho, ...)\, d\varrho}{\sqrt{(\varrho^2 - x^2)}} \tag{14}$$

we find that

$$F_c\{h(x); \xi\} = H_0\{f(\varrho); \xi\} \quad \text{if} \quad h(x) = T_2\{f(\varrho); x\}. \tag{15}$$

Similarly, if we invert the equation

$$\int_0^{\infty} J_1(\varrho\xi) \sin(x\xi)\, d\xi = \begin{cases} x\varrho^{-1}(\varrho^2 - x^2)^{-\frac{1}{2}}, & \varrho > x, \\ 0, & \varrho < x, \end{cases} \tag{16}$$

([1] p. 405), using the Hankel inversion theorem, we find that

$$\int_x^{\infty} \frac{xJ_1(\varrho\xi)\, d\varrho}{\sqrt{(\varrho^2 - x^2)}} = \xi^{-1} \sin(x\xi),$$

from which we derive the relation

$$H_1\{j(\varrho); \xi\} = \xi^{-1} F_s\{f(x); \xi\}, \quad \text{if} \quad j(\varrho) = \varrho^{-1} T_1\{xf(x); \varrho\}. \tag{17}$$

Alternatively, using the Fourier sine inversion theorem ([2], p. 18), we find that

$$F_s\{k(x); \xi\} = H_1\{f(\varrho); \xi\}, \quad \text{if} \quad k(x) = xT_2\{\varrho^{-1} f(\varrho); x\}. \tag{18}$$

We can write the relations (13), (15), (17) and (18) in the inverse forms

$$\begin{cases} f(x) = F_c^{-1}[\xi H_0\{g(\varrho); \xi] = F_s^{-1}[\xi H_1\{j(\varrho); \xi\}], \\ f(\varrho) = H_0^{-1}[F_c\{h(x); \xi] = H_1^{-1}[F_s\{k(x); \xi\}]. \end{cases} \tag{19}$$

*) Because of their obvious relation to Abel integral equations, we might call T_1 and T_2 (introduced below), *Abel transforms* of the first and second kinds, respectively.

If we write

$$T_1\{\varphi(x,\ldots);\varrho\} = \bar{\varphi}(\varrho,\ldots), \tag{20a}$$

then we can regard the defining relation (13) as an integral equation of Abel type, whose solution will give $\varphi(x,\ldots)$ in terms of $\bar{\varphi}(\varrho,\ldots)$ and hence will give the form of the operator T_1^{-1}. In this way, we find that

$$\varphi(x,\ldots) = T_1^{-1}\{\bar{\varphi}(\varrho,\ldots);x\} = \frac{2}{\pi}\frac{\partial}{\partial x}\int_0^x \frac{t\bar{\varphi}(t,\ldots)\,dt}{\sqrt{(x^2-t^2)}} = \frac{\partial}{\partial x}T_1\{t\bar{\varphi}(t,\ldots),x\}. \tag{20b}$$

Similarly, if

$$T_2\{\psi(\varrho,\ldots);x\} = \bar{\psi}(x,\ldots), \tag{21a}$$

then

$$\psi(\varrho,\ldots) = T_2^{-1}\{\bar{\psi}(x,\ldots);\varrho\} = -\frac{2}{\pi\varrho}\frac{\partial}{\partial\varrho}\int_\varrho^\infty \frac{x\bar{\psi}(x,\ldots)\,dx}{\sqrt{(x^2-\varrho^2)}} = \tag{21b}$$

$$= -\frac{1}{\varrho}\frac{\partial}{\partial\varrho}T_2\{\bar{\psi}(x,\ldots),\varrho\}.$$

3. Relations among multiple transforms

We can restate the results (13) and (17) in terms of the double transforms P_s, P_c, A_s, by defining the functions

$$g(\varrho,z) = T_1\{f(x,z);\varrho\}, \quad j(\varrho,z) = \varrho^{-1}T_1\{xf(x,z);\varrho\}. \tag{22}$$

We then have the pair of relations

$$P_c\{f(x,z);\xi,\zeta\} = \xi A_0\{g(\varrho,z);\xi,\zeta\}, \quad P_s\{f(x,z);\xi,\zeta\} = \xi A_1\{j(\varrho,z);\xi,\zeta\}. \tag{23}$$

Similarly, if we define the functions $h(x,z)$ and $k(x,z)$ by the pair of equations

$$h(x,z) = T_2\{f(\varrho,z);x\}, \quad k(x,z) = xT_2\{\varrho^{-1}f(\varrho,z);x\}, \tag{24}$$

we can write the relations (15) and (18) in the forms

$$A_0\{f(\varrho,z);\xi,\zeta\} = P_c\{h(x,z);\xi,\zeta\}, \quad A_1\{f(\varrho,z);\xi,\zeta\} = P_s\{k(x,z);\xi,\zeta\}. \tag{25}$$

There are precisely similar results for the triple transforms defined by equations (8)—(10). For example, equations (13) and (17) imply, respectively, that

$$D_0\{g(\varrho,z,t);\xi,\zeta,\omega\} = \xi^{-1}D_c\{f(x,z,t);\xi,\zeta,\omega\} \quad \text{if} \quad g(\varrho,z,t) = T_1\{f(x,z,t);\varrho\}, \tag{26}$$

and that

$$D_1\{j(\varrho,z,t);\xi,\zeta,\omega\} = \xi^{-1}D_s\{f(x,z,t);\xi,\zeta,\omega\} \quad \text{if} \quad j(\varrho,z,t) = \varrho^{-1}T_1\{xf(x,z,t);\varrho\}. \tag{27}$$

Similarly, from equations (15) and (18) we derive the relations

$$(28)\quad D_c\{h(x,z,t);\xi,\zeta,\omega\} = \\ = D_0\{f(\varrho,z,t);\xi,\zeta,\omega\}, \quad \text{if} \quad h(x,z,t) = T_2\{f(\varrho,z,t);x\}.$$

$$(29)\quad D_s\{k(x,z,t);\xi,\zeta,\omega\} = \\ = D_1\{f(\varrho,z,t);\xi,\zeta,\omega\}, \quad \text{if} \quad k(x,z,t) = xT_2\{\varrho^{-1}f(\varrho,z,t);x\}.$$

4. Related formulae

An important problem we wish to consider is that of determining the relation between two functions $f_2(x,z)$ and $f_3(\varrho,z)$, which are defined in terms of two prescribed functions $p_2(x)$ and $p_3(\varrho)$ by the formulae

$$(30)\quad F_c\{f_2(x,z);\xi\} = \theta(\xi,z)\,F_c\{p_2(x),\xi\}, \quad H_0\{f_3(\varrho,z);\xi\} = \theta(\xi,z)\,H_0\{p_3(\varrho);\xi\}.$$

Now, if we choose $p_2(x)$ in such a way that

$$(31)\qquad F_c\{p_2(x);\xi\} = H_0\{p_3(\varrho);\xi\},$$

we see that

$$(32)\qquad F_c\{f_2(x,z);\xi\} = H_0\{f_3(\varrho,z);\xi\}.$$

Hence, by equation (15), we see that

$$(33)\qquad p_2(x) = T_2\{p_3(\varrho);x\}.$$

If the solution corresponding to this value of $p_2(x)$ is $f_2(x,z)$, then we deduce that the value of $f_3(\varrho,z)$ corresponding to $p_3(\varrho)$ is given by the relation

$$(34)\qquad f_3(\varrho,z) = T_2^{-1}\{f_2(x,z);\varrho\}.$$

Similarly, if $g_2(x,z)$ and $g_3(\varrho,z)$ are related through the equations

$$(35)\quad F_s\{g_2(x,z);\xi\} = \varphi(\xi,z)\,F_c\{p_2(x),\xi\}, \quad H_1\{g_3(\varrho,z)\} = \varphi(\xi,z)\,H_0\{p_3(\varrho),\xi\},$$

we deduce, from equation (18), that if $p_2(x)$ and $p_3(\varrho)$ are related through the equation (33), then

$$(36)\qquad g_3(\varrho,z) = \varrho T_2^{-1}\{x^{-1}g_2(x,z);\xi\}.$$

We encounter a similar problem if we have relations of the type

$$(37)\qquad \begin{cases} P_c\{f_2(x,z);\xi,\zeta\} = \theta(\xi,\zeta)\,P_c\{p_2(x,z);\xi,\zeta\}, \\ A_0\{f_3(\varrho,z);\xi,\zeta\} = \theta(\xi,\zeta)\,A_0\{p_3(\varrho,z);\xi,\zeta\}. \end{cases}$$

If we choose $p_2(x,z)$, $p_3(\varrho,z)$ to be such that

$$P_c\{p_2(x,z);\xi\ \zeta\} = A_0\{p_3(\varrho,z);\xi\ \zeta\}$$

then

$$P_c\{f_2(x,z);\xi,\zeta\} = A_0\{f_3(\varrho,z);\xi,\zeta\}$$

and it follows from equations (24), (25) that these equations are equivalent to the relations

$$p_2(x, z) = T_2\{p_3(\varrho, z); x\}, \qquad f_3(\varrho, z) = T_2^{-1}\{f_2(x, z); \varrho\}. \tag{38}$$

Similarly, if we have relations of the type

$$\begin{cases} P_s\{g_2(x, z); \xi, \zeta\} = \varphi(\xi, \zeta)\, P_c\{p_2(x, z); \xi, \zeta\}, \\ A_1\{g_2(\varrho, z); \xi, \zeta\} = \varphi(\xi, \zeta)\, A_0\{p_3(\varrho, z); \xi, \zeta\}, \end{cases} \tag{39}$$

where $p_2(x, z)$ and $p_3(\varrho, z)$ are related through the first of equations (38), it follows that

$$g_3(\varrho, z) = \varrho T_2^{-1}\{x^{-1} g_2(x, z)\}. \tag{40}$$

There are similar results for the "dynamical" transforms. If two functions $f_2(x, z, t)$ and $f_3(\varrho, z, t)$ are related through the equations

$$\begin{cases} D_c\{f_2(x, z, t); \xi, \zeta, \omega\} = \varphi(\xi, \zeta, \omega)\, D_c\{p_2(x, z, t); \xi, \zeta, \omega\}, \\ D_0\{f_3(\varrho, z, t); \xi, \zeta, \omega\} = \varphi(\xi, \zeta, \omega)\, D_0\{p_3(\varrho, z, t); \xi, \zeta, \omega\}, \end{cases} \tag{41}$$

then if $p_2(x, z, t)$ is chosen to be such that

$$D_c\{p_2(x, z, t); \xi, \zeta, \omega\} = D_0\{p_3(\varrho, z, t); \xi, \zeta, \omega\},$$

we must have

$$D_c\{f_2(x, z, t); \xi, \zeta, \omega\} = D_0\{f_3(\varrho, z, t); \xi, \zeta, \omega\};$$

i.e. if

$$p_2(x, z, t) = T_2\{p_3(\varrho, z, t); x\} \tag{42a}$$

then

$$f_3(\varrho, z, t) = T_2^{-1}\{f_2(x, z, t); \varrho\}. \tag{42b}$$

Finally, if the relation between $g_2(x, z, t)$ and $g_3(\varrho, z, t)$ is

$$\begin{cases} D_s\{g_2(x, z, t); \xi, \zeta, \omega\} = \varphi(\xi, \zeta, \omega)\, D_c\{p_2(x, z, t); \xi, \zeta, \omega\}, \\ D_1\{g_3(x, z, t); \xi, \zeta, \omega\} = \varphi(\xi, \zeta, \omega)\, D_0\{p_3(\varrho, z, t); \xi, \zeta, \omega\}, \end{cases} \tag{43}$$

then if $p_2(x, z, t)$ is chosen to have the value given by equation (42a),

$$g_3(\varrho, z\ t) = \varrho T_2^{-1}\{x^{-1} g_2(x, z, t); \varrho\}. \tag{44}$$

5. Applications to potential theory

To illustrate the use of these formulae we shall consider two simple problems in potential theory *). Suppose that we have solved the following simple two-dimensional problem in potential theory:

*) Applications of these results to static and dynamic problems in elasticity will be considered in a forthcoming paper by the author and Z. Olesiak.

Find the solution $f_2(x, z)$ *of Laplace's equation*

$$\frac{\partial^2 f_2}{\partial x^2} + \frac{\partial^2 f_2}{\partial z^2} = 0,$$

in the positive half-plane $z \geqslant 0$, *satisfying the conditions* $f_2(x, 0) = p_2(x)$, $f_2(x, z) \to 0$ *as* $(x^2 + z^2)^{1/2} \to \infty$

and that we wish to deduce the solution of the following axially-symmetric problem:

Find the solution $f_3(\varrho, z)$ *of Laplace's equation*

$$\frac{\partial^2 f_3}{\partial \varrho^2} + \frac{1}{\varrho}\frac{\partial f_3}{\partial \varrho} + \frac{\partial^2 f_3}{\partial z^2} = 0,$$

in the half-space $z \geqslant 0$, *satisfying the conditions* $f_3(\varrho, 0) = p_3(\varrho)$, $f_3(\varrho, z) \to 0$ *as* $(\varrho^2 + z^2)^{1/2} \to \infty$.

It is easily shown that, in the special case in which $p_2(x)$ is an *even* function of x,

$$F_c\{f_2(x, z); \xi\} = e^{-\xi z} F_c\{p_2(x); \xi\},$$

and that the solution of the axially-symmetric problem satisfies the relation

$$H_0\{f_3(\varrho, z); \xi\} = e^{-\xi z} H_0\{p_3(\varrho); \xi\}.$$

This is precisely the situation envisaged in equations (30) — with $\theta(\xi, z) = e^{-\xi z}$. Hence it follows from equations (33) and (34) that the solution of the axially-symmetric problem is given by the formula

$$f_3(\varrho, z) = T_2^{-1}\{f_2(x, z); \varrho\} = -\frac{2}{\pi\varrho}\frac{\partial}{\partial \varrho}\int_{\varrho}^{\infty} \frac{x f_2(x, z)}{\sqrt{(x^2 - \varrho^2)}}\, dx, \tag{45}$$

where $f_2(x, z)$ is the solution of the two-dimensional problem corresponding to the boundary value

$$p_2(x) = T_2\{p_3(\varrho); x\} = \sqrt{\frac{2}{\pi}}\int_{x}^{\infty} \frac{\varrho p_3(\varrho)\, d\varrho}{\sqrt{(\varrho^2 - x^2)}}. \tag{46}$$

Finally, let us consider the two-dimensional problem in which $f_2(x, 0) = p_2^{(1)}(x)$ $f_2(x, a) = p_2^{(2)}(x)$, are the boundary conditions in the thick strip $0 \leqslant x \leqslant a$, and find its relationship to the corresponding three-dimensional problem for the thick plate $0 \leqslant z \leqslant a$ subject to the boundary conditions $f_3(\varrho, 0) = p_3^{(1)}(\varrho)$, $f_3(\varrho, a) = p_3^{(2)}(\varrho)$. It is easily shown that, for these conditions,

$$F_c\{f_2(x, z); \xi\} = \frac{\sinh \xi(a - z)}{\sinh \xi a} F_c\{p_2^{(1)}(x); \xi\} + \frac{\sinh \xi z}{\sinh \xi a} F_c\{p_2^{(2)}(x); \xi\},$$

$$H_0\{f_3(\varrho, z); \xi\} = \frac{\sinh \xi(a - z)}{\sinh \xi a} H_0\{p_3^{(1)}(\varrho); \xi\} + \frac{\sinh \xi z}{\sinh \xi a} H_0\{p_3^{(2)}(\varrho); \xi\}.$$

Hence if we choose

$$F_c\{p_1^{(1)}(x);\xi\} = H_0\{p_1^{(1)}(\varrho);\xi\}, \qquad F_c\{p_2^{(2)}(x);\xi\} = H_0\{p_2^{(2)}(\varrho);\xi\},$$

we have

$$H_0\{f_3(\varrho, z);\xi\} = F_c\{f_2(x, z);\xi\}.$$

Thus the solution of the axially-symmetric problem is given by equation (45) where $f_2(x, z)$ is the solution of Laplace's equation in the strip $0 \leqslant z \leqslant a$ satisfying the boundary conditions

$$f_2(x, 0) = \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{\varrho p_3^{(1)}(\varrho)\, d\varrho}{\sqrt{(\varrho^2 - x^2)}}, \qquad f_2(x, a) = \sqrt{\frac{2}{\pi}} \int_x^\infty \frac{\varrho p_3^{(2)}(\varrho)\, d\varrho}{\sqrt{(\varrho^2 - x^2)}}.$$

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REFERENCES

[1] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, 1944, 2nd. edition.

[2] I. N. Sneddon, *Fourier transforms*, McGraw-Hill Book Co., New York, 1951.

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MATHEMATICS

Concerning Closures of Images of the Reals

by

R. D. ANDERSON and E. H. CONNELL

Presented by K. KURATOWSKI on September 14, 1961

Continuous images of the space of real numbers arise naturally in many areas of mathematics. In this paper, a study is made of the dimension of closures of images of the reals and of one-dimensional sets in general. This study is the result of an unsuccessful attempt by the authors to construct an infinite dimensional continuum containing no non-degenerate finite dimensional subcontinuum.

Theorem 1 shows that an irrational wrapping of the reals on a torus is not sufficient for certain purposes; Theorem 2 shows that there is a mapping of a ray into a square (or torus) which would "suffice"; and Theorem 3 shows that there can be no generalization of Theorem 2 to any dimension greater than or equal to 2.

1. Notation: The cross product of a torus and an interval will be denoted by the cube $A = \{(x, y, z): 0 \leqslant x, y, z \leqslant 1\}$ with the following identifications: $(x, 0, z) = (x, 1, z)$ and $(0, y, z) = (1, y, z)$. The torus will be denoted by $T = T \times 0 = \{(x, y, z) \in T \times I : z = 0\}$ and the unit interval will be denoted by $I = [0, 1]$. The non-negative reals will be denoted by $R = [0, \infty]$.

THEOREM 1. *Let $f : R \to T$ be the infinite wrapping given by the formula $f(x) = (x, \alpha x)$ (mod 1) where α is a positive irrational number. Let $F : R \times I \to T \times I$ be defined by $F(r, t) = (f(r), t)$. Then there exists an unbounded closed connected set $C \subset (R \times I)$ such that $\mathrm{Cl}(F(C))$ is one-dimensional.*

Proof. The idea of the proof is to construct in $T \times I$ a one-dimensional continuum whose inverse under F contains an unbounded component.

Let F_1 F_2, F_3 be three faces of A, F_1 in the xz-plane, F_2 in the yz-plane, and F_3 in the xy-plane. Let $a_1, a_2, a_3, \ldots$ be a countable dense subset of $F_1 \cup F_2 \cup F_3$ such that no a_i belongs to an edge of A. If $a_i \in F_1$ or F_2, let L_i be the straight line in F_3 which contains a_i and is perpendicular to F_1 or, respectively, F_2. If $a_i \in F_3$, let L_i be the union of two straight lines, each parallel to $y = -z$, $x = 0$ with one containing a_i and the other containing $(a_i) + (0, 1, 0)$. It is necessary to take two lines in this case because these lines have been tilted down and will run through the bottom face, F_2, of A and thus under the identifications made must reappear at the top face. Let the open cylinder of radius ε (or in the third case, two open cylinders) centered around L_i be $R(L_i, \varepsilon) = \{(x, y, z): d((x, y, z), L_i) < \varepsilon\}$.

Notice that $F^{-1}(L_i)$ is a countable set of points and that the distance between any two is $\geqslant 1$. If $d > 0$, there exists $\varepsilon > 0$ such that for any positive integer i, $F^{-1}(R(L_i, \varepsilon))$ is a countable collection of disjoint open 2-cells, each with diameter $< d$.

Let $\varepsilon_n > 0$ such that $F^{-1}(R(L_n, \varepsilon_n))$ is a collection of disjoint open 2-cells, each with diameter $< \frac{1}{2^{n+3}}$. Let $Q = \bigcup_{n=1}^{\infty} F^{-1}(R(L_n, \varepsilon_n))$. Now $\mathrm{Cl}[F(R \times I - Q)] \subset \subset [T \times I - \bigcup_{n=1}^{\infty} R_n(L_n, \varepsilon_n)]$ which is a closed one-dimensional set. (A suggestion of an argument for this is given in Section 3).

The only remaining step is to prove that $R \times I - Q$ contains an unbounded closed connected set. Now $G_k = \bigcup_{n=1}^{k} F^{-1}[\mathrm{Cl}(R(L_n, \varepsilon_n))]$ can contain no continuum of diameter $> \frac{1}{2}$ because it would then contain a continuum B of diameter $= \frac{1}{2}$. However, B could contain at most one 2-cell from any given $F^{-1}[\mathrm{Cl}(R(L_n, \varepsilon_n))]$ and thus the total diameter of B must be $< \frac{1}{4}$. Hence, G_k cannot separate $0 \times I$ from $k \times I$ in $R \times I$ because if G_k separates, some component of G_k must separate, and this is impossible.

Therefore there exists a continuum $C_k \subset R \times I - G_k$ with C_k intersecting $0 \times I$ and $k \times I$. In the one point compactification of $R \times I$, $\{C_k\}$ contains a convergent subsequence. Its limit, minus the point at infinity, is an unbounded closed set L, and L must be contained in $R \times I - Q$ and must intersect $0 \times I$. Let C be a component of L intersecting $0 \times I$. Then C must be unbounded, otherwise $L + \infty$ would not be connected.

2. Notation. If E is a collection of open sets, a chain from E of length n is a finite sequence $O_1, O_2, \ldots, O_n$ where each $O_i \varepsilon E$ and $O_i \cap O_{i+1} \neq \emptyset$. Any type of repetition is allowed and thus there may be more elements in the chain than in E itself. A function f defined on $[a, b]$ is said to run through a chain $O_1, O_2, \ldots, O_n$ provided there exists numbers $c_0, c_1, \ldots, c_n$ with $a = c_0 \leqslant c_1 \leqslant \ldots \leqslant c_n = b$ and $f([c_i, c_{i+1}]) \subset O_{i+1}$. S will denote the unit square, $S = \{(x, y): 0 \leqslant x, y \leqslant 1\}$.

THEOREM 2. *There exists a continuous function f: $R \to S$, the unit square, such that if F: $R \times I \to S \times I$ is defined by $F(r, t) = (f(r), t)$ and C is an unbounded connected subset of $R \times I$, then* $\mathrm{Cl}(F(C))$ *has dimension* $\geqslant 2$.

Proof. For each positive integer n let $E_n = O_{11}, O_{12}, \ldots, O_{1m_n}$ be an open covering of S with each open set connected and of diameter less than $1/n$. Let f map R into S and have the following property: given any chain of length n composed of open sets from $\bigcup_{i=1}^{n} E_i$, there exists an interval $[a, b] \subset [n - 1, n]$ such that $f \mid [a, b]$ runs through it. The function f will then have this property: if E is any chain of open sets of length k, there exists an interval $[c, d]$ such that $f \mid [c, d]$ runs through E. Also, if $N > 0$, $[c, d]$ may be selected so that $c > N$.

Now suppose C is a closed and connected, unbounded subset of $R \times I$. It will be shown that $\mathrm{Cl}(F(C))$ must separate the two faces $(S, 0)$ and $(S, 1)$ and thus must have dimension at least 2.

Suppose $\mathrm{Cl}(F(C))$ does not separate these two faces. Then there exists a chain $V_1, V_2, \ldots, V_n$ of open cubes (with edges parallel to the axes) such that $V_1 \cap (S, 0) \neq \emptyset$ and $V_n \cap (S, 1) \neq \emptyset$ and $V_i \cap \mathrm{Cl}(F(C)) = \emptyset$ for $i = 1, 2, \ldots, n$. The projection of the V_i's on $(S, 0)$ will form a chain of length n of open squares. There exists an interval $[c, d]$ such that $f|[c, d]$ runs through this new chain in $(S, 0)$. Furthermore, $[c, d]$ may be selected so that some point of C has first coordinate less than c.

Now there exists an arc $A \subset [c, d] \times I \subset R \times I$ from $(c, 0)$ to $(b, 1)$ which is composed of n straight line intervals such that $F|A$ runs through $V_1, V_2, \ldots, V_n$. Thus $C \cap A = \emptyset$, but this is a contradiction because C is connected and A separates $R \times I$ into two sets each intersecting C. This completes the proof.

3. In this section we prove the following

THEOREM 3. *Let R be the ray $[0, \infty)$ and f be a mapping of R into S (the unit square in the xy-plane). Let F be the mapping of $R \times I^2$ into $S \times I^2$ such that $F(r, p) = (f(r), p)$ Then there exists an unbounded closed connected set C in $R \times I^2$ such that $\mathrm{Cl}(F(C))$ is at most 1-dimensional.*

The methods we are to use in the proof of Theorem 3 are somewhat similar to those of the proof of Theorem 1. Before beginning the actual proof we introduce some notation and recall some facts of elementary dimension theory.

Let M denote the unit 4-cube in E^4. Let $\tilde{M}$ be the set of all points of M with at most one rational coordinate properly between 0 and 1. The set M is 1-dimensional since (1) M is connected and (2) M can be covered by small sets with 0-dimensional boundaries. To see the latter note that if K is any hyperplane defined by an equation of the form $x_i = r$ for r rational and between 0 and 1, then $K \cap \tilde{M}$ is 0-dimensional. Clearly any non-degenerate continuum in $\tilde{M}$ is also 1-dimensional.

A θ-sequential operation is a process for exhibiting a certain type of closed and connected non-degenerate subset M_0 of M. M_0 will equal $\bigcap_{i \geq 1} M_i$. Let $\{V_i\}$ be the set of two-dimensional planes defined by equations of the form $x_j = r_1$, $x_k = r_2$ where $1 \leq j, k \leq 4$, $j \neq k$ and r_1 and r_2 are rationals properly between 0 and 1. Clearly $M - \bigcup_{i \geq 1} V_i$ is $\tilde{M}$. Let U_1 be a rectangular neighborhood of V_1 defined by inequalities of the form $0 < r_1 - a_1 < x_j < r_1 + a_1 < 1$, $0 < r_2 - a_2 < x_k < r_2 + a_2 < 1$ where a_1 and a_2 are positive and irrational. Let M_1 be $M - U_1$. Let i_2 be the least integer for which $V_{i_2} \not\subset U_1$. Let U_{i_2} be a similar neighborhood of V_{i_2} with $U_1 \cap U_{i_2} = \emptyset$ if V_1 and V_{i_2} are parallel. Let $M_2 = M_1 - U_{i_2}$. Let i_3 be the least integer for which $V_{i_3} \not\subset U_1 \cup U_{i_2}$. Let U_{i_3} and M_3 be similarly defined. Clearly we may induct. This inductive operation is called a θ-operation on M.

Proof of Theorem 3. We consider S as the unit square $0 \leq x \leq 1$, $0 \leq y \leq 1$, $z = 0$, $w = 0$ in 4-space with $S \times I^2$ as the unit 4-cube. Let M be a geometric 4-cube containing $S \times I^2$ such that any 2-plane parallel to a 2-face of M intersects any square $s \times I^2$ in at most a single point. We consider M as being oriented with respect to coordinates (x_1', x_2', x_3', x_4') and we may apply a sequential θ-operation to M. We choose to do this in such a way that the "widths" of the slices U_{i_j} are

restricted by the nature of the mapping F. For any r, $0 \leqslant r < \infty$, we let F_r denote F restricted to $r \times I_2$.

Let $\{V_i\}$ be a sequence of the 2-dimensional hyperplanes as above. Now we note that V_1 intersects any square $F(r \times I^2)$ in at most one point and thus, for any r, $F_r^{-1}(V_1)$ is not more than a single point. Hence we may delete a rectangular neighborhood U_1 of V_1 from M to form M_1 such that, for any $r \geqslant 0$, $F_r^{-1}(M_1)$ is all of $r \times I^2$ except for a small open set, say of diameter less than 1/4. Then clearly $F^{-1}(M_1)$ contains an unbounded component containing almost all of $R \times I^2$. The component which contains most (or all) of $0 \times I^2$ must contain most of each set $r \times I^2$, for $r > 0$, since there can be neither a smallest r not having this property nor a largest one having it.

Now $F_r^{-1}(V_{i_2})$ (with V_{i_2} as in the 0-sequential operation above) is not more than a single point and thus there exists a rectangular neighborhood U_{i_2} of V_{i_2} of such small width that $M_1 - U_{i_2} (= M_2)$ has the property that $F_r^{-1}(M_2)$ is all of $r \times I^2$ except for two small open sets the sum of whose diameters is less than $1/4 + 1/8$. As above $F^{-1}(M_2)$ contains a component containing almost all of $R \times I^2$. The process clearly inducts with the set $M_0 = \bigcap_{i \geqslant 1} M_i$ being such that $F^{-1}(M_0)$ contains an unbounded component C with $\mathrm{Cl}(F(C))$ at most 1-dimensional.

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MATHEMATIQUE

Problème de Cauchy pour un système parabolique dont les coeffcients dépendent de fonctions inconnues

par

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Le présent travail est basé sur les résultats de W. Pogorzelski publiés dans [1]—[3], où l'auteur a déterminé une matrice des solutions fondamentales du système parabolique et étudié le problème de Cauchy pour le système parabolique quasi-linéaire.

Soit le système de $N \geqslant 1$ équations aux dérivées partielles d'ordre $M \geqslant 2$

$$(1) \qquad \hat{\Psi}^{(\alpha)}_{(u)} = \sum_{1 \leqslant j \leqslant N}^{0 \leqslant k_1 + \ldots + k_n \leqslant M} A^{k_1 \ldots k_n}_{\alpha j \ldots}(X, t, u) \frac{\partial^{k_1 + \ldots k_n} u_j}{\partial x_1^{k_1} \ldots x_n^{k_n}} - \frac{\partial u_\alpha}{\partial t} = 0, \qquad (\alpha = 1, \ldots, N),$$

où u désigne le système de N fonctions inconnues

$$(2) \qquad u = [u_1(X, t), \ldots, u_N(X, t)].$$

On admet que

$$\frac{\partial^0 u_j}{\partial x_1^0 \ldots \partial x_n^0} = u_j$$

Nous admettons les hypothèses suivantes.

HYPOTHÈSE 1. *Les coefficients* $A^{k_1 \ldots k_n}_{\alpha j}(X, t, u)$ *sont des fonctions définies et bornées dans la région*

$$(3) \qquad X \in E, \quad 0 \leqslant t \leqslant T, \quad -\infty < u_j < \infty; \quad (j = 1, \ldots, N),$$

où E *est un espace euclidien de points* $X(x_1, \ldots, x_n)$.

Ces coefficients vérifient les conditions de Hölder de la forme

$$(4) \qquad |A^{k_1 \ldots k_n}_{\alpha j}(X, t, u) - A^{k_1 \ldots k_n}_{\alpha j}(X', t', u')| < k_A \left\{ |XX'|^h + |t - t'|^{h'} + t^\gamma \exp[-b|XX_0|] \sum_{j=1}^{N} |u_j - u'_j|^{h''} \right\},$$

si

$$k_1 + \ldots + k_n = M$$

et

$$(5)\quad |A_{\alpha j}^{k_1 \dots k_n}(X, t, u) - A_{\alpha j}^{k_1 \dots k_n}(X', t, u')| < \\ < k'_A \left\{ |XX'|^h + t^\gamma \exp[-b|XX_0|] \sum_{j=1}^{N} |u_j - u'_j|^{h''} \right\},$$

si

$$0 \leqslant k_1 + \dots + k_n < M, \qquad (\alpha, j = 1, \dots, N),$$

où les exposants h, h', h'' sont des constantes positives non supérieures à l'unité, les coefficients k_A, k'_A sont positifs, γ est une constante positive non supérieure à l'unité, b — une constante positive, X_0 — un point fixé dans l'espace E; $|XX'|$ désigne la distance euclidienne entre les points X et X' dans l'espace E.

Nous admettons pour les constantes données les inégalités suivantes

$$(5')\quad \begin{cases} \max(h, h') < h'' \leqslant 1; \qquad h'/h'' < \min(n/M, 1) \\ \max\left(h'/h'', 1 - \dfrac{h^*}{M}\right) < \gamma < h'' \leqslant 1, \quad \text{où } h^* = \min(h, Mh') \end{cases}$$

En outre, les coefficients $A_{\alpha j}^{k_1 \dots k_n}$ pour $0 \leqslant k_1 + \dots + k_n < M$ sont uniformément continus par rapport à la variable t.

Hypothèse II. *Conformément à la définition du système parabolique donnée par Petrovsky* [4], *l'équation*

$$(6)\quad \det_{\alpha, j} \Big| \sum_{k_1 + \dots + k_n = M} A_{\alpha j}^{k_1 \dots k_n}(X, t, u)\,(is_1)^{k_1} \dots (is_n)^{k_n} - \lambda \delta_\alpha^j \Big| = 0$$

possède des racines dont toutes les parties réelles sont inférieures à un nombre négatif fixé

$$(7)\quad \operatorname{Re}(\lambda) < -\delta < 0$$

quelles que soient les valeurs des variables réelles $s_1, \dots, s_n$ *vérifiant la relation*

$$(8)\quad s_1^2 + \dots + s_n^2 = 1$$

et pour tous les points de la région (3); δ_α^j *est un symbole de Kronecker.*

En vertu des hypothèses admises, le degré M du système (1) est pair.

Nous posons le problème de la détermination d'un système de fonctions

$$(9)\quad u = [u_1(X, t), \dots, u_N(X, t)]$$

qui

1) vérifie le système (1) en tout point $X \in E$ pour $0 < t < T$

2) vérifie la condition initiale

$$(10)\quad \lim_{t \to 0} u_\alpha(X, t) = f_\alpha(X), \qquad (\alpha = 1, \dots, N)$$

en tout point $X \in E$.

Hypothèse III. *On admet que les fonctions données* $f_\alpha(X)$ *sont définies et continues dans l'espace E et qu'elles vérifient les inégalités*

$$(11)\quad |f_\alpha(X)| \leqslant a_f \exp[b|XX_0|]$$

où a_j est une constante positive donnée, la constante positive b étant admise dans les hypothèses I, (4) *et* (5).

Avant de résoudre le problème posé, nous présenterons quelques formules fondamentales qui seront nécessaires dans la suite.

Soit une opération différentielle sur le système de fonctions $v = [v_1, ..., v_N]$:

$$(12)\qquad \overset{(a)}{\Psi}_{(u)}[v(X,t)] = \sum_{1\leqslant j\leqslant N}^{0\leqslant k_1+\ldots k_n\leqslant M} A_{aj}^{k_1,\ldots k_n}[X,t,u(X,t)]\,\frac{\partial^{k_1+\ldots+k_n}v_j}{\partial x_1^{k_1}\ldots\partial x_n^{k_n}} - \frac{\partial v_a}{\partial t},$$

où les coefficients dépendent des variables X, t et d'un système de fonctions $u = [u_1(X,t), \ldots, u_N(X,t)]$ vérifiant les inégalités

$$-\infty < u_j(X,t) < \infty$$

$$(13)\qquad t^{\gamma/h''}\exp\left|-\frac{b}{h''}\,XX_0\right|\,|u_j(X,t) - u_j(X',t')| < k\,[|XX'|^{\theta}+|t-t'|^{\theta'}].$$

Les constantes γ, h'', b se trouvent dans (4) et (5), k est une constante positive arbitraire. On a posé

$$(13')\qquad \theta = h/h'' < 1; \qquad \theta' = h'/h'' < \min(n/M, 1).$$

Conformément à la formule (85) du [1], la formule

$$(14)\qquad \Gamma_{\alpha\beta}^{(u)}(X,t;Y,\tau) = W_{\alpha\beta}^{Y,\tau,(u)}(X,t;Y,\tau) + \int_{\tau}^{t}\int_{E}\sum_{\gamma=1}^{N} W_{\alpha\gamma}^{\Pi,\zeta,(u)}(X,z;\Pi,\zeta)\,\Phi_{\gamma\beta}^{(u)}(\Pi,\zeta;Y,\tau)\,d\Pi\,d\zeta$$

donne une matrice $\{\Gamma_{\alpha\beta}^{(u)}\}$ des solutions fondamentales du système

$$\overset{(a)}{\Psi}_{(u)}[v(X,t)] = 0, \qquad (\alpha = 1, ..., N).$$

Les fonctions $W_{\alpha\beta}^{Z(u)}(X,t;Y,\tau)$ sont données par les formules

$$(15)\qquad W_{\alpha\beta}^{Z,\zeta,(u)}(X,t;Y,\tau) = \frac{1}{(2\pi)^n}\int_{E} w_{\alpha}^{\beta(u)}(t,\tau;Z,\zeta;S)\exp\left[i\sum_{v=1}^{n} s_v(x_v-\xi_v)\right]dS$$

où l'intégrale est étendue à l'espace E de variables réelles $S(s_1, ..., s_n)$. Les symboles $X(x_1, ..., x_n)$ et $Y(\xi_1, ..., \xi_n)$ désignent deux points arbitraires de l'espace E; Z est un point arbitrairement fixé dans E, ζ est fixé dans l'intervalle $(0, T)$. Les fonctions (dépendant de u)

$$w_1^{\beta,(u)}, w_2^{\beta,(u)}, \ldots, w_N^{\beta,(u)}, \qquad (\beta = 1, ..., N)$$

forment, pour β fixé, une solution du système d'équations différentielles ordinaires

$$(16)\qquad \frac{dw_{\alpha}^{\beta,(u)}}{dt} = \sum_{1\leqslant j\leqslant N}^{k_1+\ldots+k_n=M} A_{aj}^{k_1,\ldots k_n}[Z,\zeta;u(Z,\zeta)]\,(is_1)^{k_1}\ldots(is_n)^{k_n}\,w_j^{\beta,(u)}(t,\tau;Z,\zeta;S)$$

avec la condition initiale $w_{\alpha}^{\beta(u)} \to \delta_{\alpha}^{\beta}$, si $t \to \tau$.

Les fonctions $\Phi_{\alpha\beta}^{(u)}$ dans les formules (14) forment, pour β fixé, une solution du système d'équations intégrales de Volterra:

$$(17)\quad \Phi_{\alpha\beta}^{(u)}(X,t;Y,\tau) = \hat{\Psi}_{(u)}^{(a)}[W_j^{Y,\tau,(u)}(X,t;Y,\tau)] + \int_{\tau}^{t}\int_{E}\int\int\sum_{\gamma=1}^{N}\hat{\Psi}_{(u)}^{(a)}[W_{j\gamma}^{\Pi,\zeta,(u)}(X,t;\Pi,\zeta)]\,\Phi_{\gamma\beta}^{(u)}(\Pi,\zeta;Y,\tau)\,d\Pi\,d\zeta$$

Les équations (17) possèdent les noyaux à faibles singularités et sont résolubles par les formules classiques de Volterra.

Passons à la résolution du problème proposé.

Nous allons chercher la solution sous la forme d'un système d'intégrales généralisées de Poisson-Weierstrass

$$(18)\quad u_\alpha(X,t) = \int_{E}\int\int\sum_{j=1}^{N}\Gamma_{\alpha j}^{(u)}(X,t;Y,0)\,f_j(Y)\,dY, \qquad (\alpha = 1,\ldots,N).$$

Les formules (18) présentent un système de N équations intégrales à N fonctions inconnues $u\,[u_1,\ldots,u_N]$.

Nous allons résoudre le système (18) en appliquant le théorème topologique de J. Schauder [5], que nous reproduisons ci-dessous dans une forme concise:

„Pour toute opération continue d'un ensemble fermé et convexe, dans l'espace de Banach, en sa partie compacte, il existe au moins un point invariant".

Soit un espace fonctionnel Λ composé de tous les systèmes de fonctions réelles

$$u = [u_1(X,t),\ldots,u_N(X,t)]$$

définies et continues dans la région ouverte $[X\,\varepsilon\,E,\ 0 \leqslant t \leqslant T]$ et vérifiant l'inégalité

$$(19)\quad \sup_{\substack{X\,\varepsilon\,E\\ t\,\varepsilon\,(0,T)}}\{t^{\gamma'}\exp[-b'|XX_0|]\,|u_j|\} < \infty,\ (j = 1,\ldots,N),$$

où $\gamma' > \gamma$, γ et b sont des constantes dans les hypothèses I, (4), (5), (5'), b' est une nouvelle constante supérieure à b.

On admet pour l'espace Λ les définitions connues des opérations linéaires et la définition suivante de la norme d'un élément $u(X,t)$:

$$(20)\quad \|u(X,t)\| = \max_{j}\sup_{\substack{X\,\varepsilon\,E\\ t\,\varepsilon\,(0,T)}}\{t^{\gamma'}\exp[-b'|XX_0|]\,|u_j(X,t)|\}, \qquad (j = 1,\ldots,N).$$

La distance entre les deux points u et u^* est définie par la formule

$$(21)\quad \delta(u,u^*) = \|u - u^*\|.$$

L'espace Λ est linéaire, métrique et normé. En plus, on peut démontrer qu'il est complet. Donc Λ est un espace de Banach. Considérons maintenant dans l'espace Λ un ensemble Z de tous les systèmes $u = [u_1(X,t),\ldots,u_N(X,t)]$ de fonctions vérifiant les inégalités

$$(22)\quad \exp[-b|XX_0|]\,|u_j(X,t)| \leqslant \varrho,$$

$$(23)\quad t^{\gamma}t^{h''}\exp\left[-\frac{b}{h''}|XX_0|\right]|u_j(X,t) - u_j(X',t')| \leqslant \varkappa\,[|XX'|^{\vartheta} + |t-t'|^{\vartheta}]$$

où

$$\theta = \frac{h}{h''} < 1, \qquad \theta' = \frac{h'}{h''} < \min(n/M, 1),$$

$\varkappa$ est un nombre positif arbitrairement fixé, ϱ est une constante positive suffisamment grande.

Conformément à la forme du système (18), transformons l'ensemble Z à l'aide des relations

$$(24) \qquad v_\alpha(X,t) = \int\int_E \int \sum_{j=1}^{N} \Gamma_{\alpha j}^{(u)}(X,t;Y,0) f_j(Y)\, dY, \qquad (\alpha = 1, \ldots, N)$$

qui à tout point $u = [u_1(X,t), \ldots, u_N(X,t)] \varepsilon Z$ font correspondre un système de fonctions $v = [v_1(X,t), \ldots, v_N(X,t)]$.

On peut démontrer que l'ensemble Z' de tous les systèmes transformés fait partie de l'ensemble Z, si les inégalités

$$(25) \qquad C(\varkappa)\, a_f \leqslant \varrho,$$

$$(26) \qquad T^{\frac{\gamma}{h''} - \theta'} a_f D(\varkappa) \leqslant \varkappa$$

sont vérifiées; $C(\varkappa)$ et $D(\varkappa)$ sont des constantes positives dépendant de coefficient $\varkappa$ dans (23). Nous rappelons que $\gamma/h'' < \theta'$, conformément à (5') et (13'). D'après la forme de (25) et (26), on voit que ces conditions seront vérifiées si la constante ϱ est choisie suffisamment grande et les constantes a_f ou T sont suffisamment petites.

On peut démontrer que la transformation définie par les relations (24) est continue dans l'espace A et que l'ensemble transformé Z' est compact.

Nous en concluons, en vertu du théorème cité de J. Schauder, qu'il existe au moins un point u^* invariant de la transformation (24), c.-à-d. une solution $[u_1^*(X,t), \ldots, u_N^*(X,t)]$ du système d'équations intégrales (18). D'après le théorème 4 du travail [3], ce système satisfait à l'équation (1) en tout point X de l'espace E et pour t à l'intérieur de l'intervalle (O, T).

D'après le théorème 3 du travail [3], le système trouvé satisfait à la condition initiale (10).

Nous pouvons donc énoncer le théorème suivant:

THÉORÈME. *Si les hypothèses* I, II, III *sont vérifiées, si, en outre, l'une des constantes* a_f *ou* T *est suffisament petite pour que l'inégalité* (26) *soit satisfaite, alors il existe au moins un système de fonctions* $u_1(X,t), \ldots, u_N(X,t)$ *qui vérifie le système parabolique* (1) *donné en tout point* X *de l'espace* E *pour* $0 < t < T$ *et la condition initiale* $\lim_{t \to 0} u_\alpha(X,t) = f_\alpha(X)$.

Les considérations plus détaillées seront publiées prochainement.

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OUVRAGES CITÉS

[1] W. Pogorzelski, *Étude de la matrice des solutions fondamentales du système parabolique d'équations aux dérivées partielles*, Ricerche di Matem., **7** (1958).

[2] — , *Propriétés des solutions du système parabolique d'équations aux derivées partielles*, Mathem. Scand., **6** (1958).

[3] — , *Propriétés des intégrales généralisées de Poisson-Weierstrass et problème de Cauchy pour un système parabolique*, Ann. Sc. L'Éc. Norm. Sup., **75** (1959).

[4] J. Petrovsky, *Über das Cauchysche Problem für ein System linearer partieller Differentialgleichungen im Gebiete der nicht analytischen Funktionen*, Bull. de l'Univ. de Moscou, **7** (1938).

[5] J. Schauder, *Der Fixpunktsatz in Funktionalräumen*, Stud. Mathem., (1930), 171—180.

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МАТЕМАТИКА

П. И. ПЕТРОВ

КЛАССИФИКАЦИЯ РИМАНОВЫХ ПРОСТРАНСТВ ВПОЛНЕ ВЫРОЖДЕННОЙ КРИВИЗНЫ

Представлено А. МОСТОВСКИМ 29 октября 1961 г.

1. R. Deveber [1] впервые изучил римановы многообразия вырожденной кривизны M_n. Ряд предложений об этих пространствах опубликовал Isuké Satô [2]. В настоящей заметке вводится понятие характеристики названных пространств и с помощью её даётся их классификация.

2. Пусть g_{ij} — метрический тензор, а R_{hijk} — тензор кривизны пространства M_n. Положим

$$\varphi = R_{hijk}\, u^{hi}\, u^{jk}. \tag{1}$$

$$\psi = g_{hijk}\, u^{hi}\, u^{jk}, \tag{2}$$

где $u^{ij} = -u^{ji}$, $g_{hijk} = g_{ik}\, g_{hj} - g_{ij}\, g_{hk}$ — элементы второй производной матрицы $\mathfrak{G}$ от невырожденной матрицы $g = (g_{ij})$. Форма ψ неособенная, так как по теореме Sylwester-Franke:

$$|\mathfrak{G}| = |g|^{n-1}.$$

Характеристикой $[e_1\, e_2 \ldots e_N]$ пространства M_n назовём вейерштрассову характеристику пучка форм

$$\varphi - \lambda\psi. \tag{3}$$

3. Из теории элементарных делителей известно, что форма φ от $N = \dfrac{n(n-1)}{2}$ переменных u^{ij} или её матрица $\mathcal{R} = (R_{hijkl})$ обладает рангом $\varrho = 1$ тогда и только тогда, если $N-1$ и только $N-1$ элементарных делителей системы (3) будут вида λ^e. Значит, характеристики M_n могут быть выражены символами:

$$[1\underbrace{(1\,1 \ldots 1)}_{N-1}], \qquad [\underbrace{(2\,1 \ldots 1)}_{N-1}] \tag{4}$$

Существование пространств M_n с указанными характеристиками иллюстрируется примерами:

$$ds^2 = e^{2x_2}\, dx_1^2 + \sum_{i=2}^{n} \varepsilon_i\, dx_i^2 \qquad (\varepsilon_i = \pm 1), \tag{5}$$

$$ds^2 = e^{x_3}\, dx_1^2 + 2\, dx_1\, dx_2 + dx_3^2 + \sum_{i=4}^{n} dx_i^2. \tag{6}$$

В самом деле, для (5) имеем

$$\mathcal{R} - \lambda\mathfrak{G} = \left(\begin{array}{c|c} -e^{2x_2}(1+\varepsilon_2\lambda) & 0 \\ \hline 0 & -\lambda P_{N-1} \end{array}\right), \tag{7}$$

где детерминант матрицы $N-1$ порядка $P_{N-1} = (g_{hijk})$ отличен от нуля. Следовательно, дифференциальная квадратичная форма ds^2, заданная формулой (5), обладает первой из (4) характеристикой.

Аналогичные вычисления для (6) дают:

$$\mathcal{R} - \lambda\mathfrak{G} = \left(\begin{array}{ccc|c} \lambda & 0 & 0 & \\ 0 & a+2a\lambda & -\lambda & 0 \\ 0 & -\lambda & 0 & \\ \hline & 0 & & -\lambda Q_{N-3} \end{array}\right). \tag{8}$$

где Q_{N-3} — квадратная матрица $N-3$ порядка от g_{hijk} с неравным нулю определителем, а $a = -\frac{1}{2}e^{x_3}$. Отсюда заключаем, что характеристика λ — матрицы (8) действительно выражается вторым символом из (4).

4. Два эквивалентных M_n, M'_n имеют, очевидно, одну и ту же характеристику. Обратное заключение не всегда справедливо. Как это делается в алгебре, условимся отнести к одному классу все те M_n, у которых характеристики совпадают. Пользуясь введенными выше термином и соглашением, можно формулировать теорему —

Римановы пространства вполне вырожденной кривизны разбиваются по их характеристикам

$$1.\quad [1\,(\underbrace{\mathring{1}\,\mathring{1}\,\ldots\,\mathring{1}}_{N-1})], \qquad 2.\quad [(\underbrace{\mathring{2}\,\mathring{1}\,\mathring{1}\,\ldots\,\mathring{1}}_{N-1})]$$

на два непустых неэквивалентных класса.

ИНСТИТУТ МАТЕМАТИКИ, КАЗАНСКИЙ УНИВЕРСИТЕТ

ЦИТИРОВАННАЯ ЛИТЕРАТУРА

[1] R. Deveber, *Sur les espaces de Riemann à quatre dimensions à courbure totalement dégénérée*, Bull. Acad. Royale Belgique, Cl. Sci., **42** (1956), 1033—1044.

[2] Isuké Satô, *On the Riemannian manifolds of separated curvature*, Tohoku Math. Journ., Series 2, **10** (1958), 130—134.

P. I. PETROV, CLASSIFICATION DES ESPACES RIEMANNIENS À COURBURE TOTALEMENT DÉGÉNÉRÉE

L'espace V_n est dit à courbure totalement dégénérée si la forme $R_{ijkl}\,p^{ij}\,p^{kl}$, où $p^{ij} = -p^{ji}$ est de rang un (voir [1]). Dans le présent travail on a introduit l'invariant arithmétique qui est la caractéristique de la forme métrique de l'espace riemannien à courbure totalement dégénérée. Ensuite les variétés indiquées se partagent en types selon ces caractéristiques. Il existe deux classes d'espaces à courbure totalement dégénérée qui ne sont pas équivalentes entre elles. Eles se caractérisent, respectivement, par des invariants arithmétiques

$$[1\,(\underbrace{\mathring{1}\,\mathring{1}\,\ldots\,\mathring{1}}_{N-1})], \qquad [(\underbrace{\mathring{2}\,\mathring{1}\,\ldots\,\mathring{1}}_{N-1})].$$

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On the Majorization Problem in Perturbation Theory

by

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1. In this work we discuss the problems of the Feynman diagram's majorization with the help of general parametric representation for the scattering amplitude in the perturbation theory. We consider the most general case of the diagram with an arbitrary structure of vertices and as a result we obtain a new criterion of majorization dual to that of K. Symanzik [2].

Let us demonstrate some theorems which enable us to construct in a very simple manner the primitive diagrams for any process.

2. Let G denote a strongly connected Feynman diagram with L internal lines, V vertices and V external lines.

We assume all lines to be scalar and carry the moments q_k and masses $m_k > 0$. The momentum conservation law in every vertex of the diagram is as follows.

$$\sum_{k=1}^{L} e_k^r q_k = p_r \qquad (r = 1, \ldots, V). \tag{2.1}$$

Coefficients e_k^r are elements of the incidence matrix of the diagram

$$e_k^r = \begin{cases} +1 \text{ if line } k \text{ goes out from vertex } r, \\ -1 \text{ if line } k \text{ comes into vertex } r, \\ 0 \text{ if avoiding it.} \end{cases} \tag{2.2}$$

The vanishing of total momentum is a necessary and sufficient condition for consistence of the set Eqs. (2.1).

$$\sum_{r=1}^{V} p_r = 0. \tag{2.3}$$

The rank of matrix $e(G) = \|e_k^r\|$ is equal to $V - 1$. Without loss of generality we can assume that the general minor of $V - 1$ degree is not vanishing.

$$\det e_{V-1}(G) = \begin{vmatrix} e_1^1, & \ldots, & e_{V-1}^1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ e_1^{V-1}, & \ldots, & e_{V-1}^{V-1} \end{vmatrix} \neq 0. \tag{2.4}$$

It may be demonstrated [6] that this determinant may be equal to $+1$ or -1, whereas co-factors of all matrix elements can be equal only to ± 1 or 0.

We shall consider the set of equations

$$\left.\begin{aligned} e_1^1 q_1 + \ldots + e_L^1 q_L &= p_1 \\ &\cdots\cdots\cdots\cdots \\ e_1^{V-1} q_1 + \ldots + e_L^{V-1} q_L &= p_{V-1} \\ q_V &= q_V \\ &\cdots \\ q_L &= q_L \end{aligned}\right\} \quad \text{or} \quad \Lambda q = r, \tag{2.5}$$

where

$$q = \begin{bmatrix} q_1 \\ \vdots \\ q_{V-1} \\ q_V \\ \vdots \\ q_L \end{bmatrix}; \quad r = \begin{bmatrix} p_1 \\ \vdots \\ p_{V-1} \\ q_V \\ \vdots \\ q_L \end{bmatrix}; \quad \Lambda = \begin{bmatrix} e_1^1, & \ldots, e_{V-1}^1, & e_V^1, & \ldots, e_L^1 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ e_1^{V-1}, & \ldots, e_{V-1}^{V-1}, & e_V^{V-1}, & \ldots, e_L^{V-1} \\ 0, & \ldots, 0, & 1, & \ldots, 0 \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ \cdots\cdots & \cdots\cdots & \cdots\cdots & \cdots\cdots \\ 0, & \ldots, 0, & 0, & \ldots, 1 \end{bmatrix} \tag{2.6}$$

The solution of these equations is of the form

$$q = \Lambda^{-1} r \tag{2.7}$$

where

$$\Lambda^{-1} = \begin{bmatrix} e_{V-1}^{-1}, & d \\ 0, & E \end{bmatrix} \tag{2.8}$$

$$(e_{V-1}^{-1})_{ik} = \frac{[e_i^k]}{\det e_{V-1}} \qquad (i, k = 1, \ldots, V-1) \tag{2.9}$$

$$d_{rr'} = -\sum_{i=1}^{V-1} \frac{[e_r^i]\, e_{r'}^i}{\det e_{V-1}} \qquad (r, r' = V, \ldots, L) \tag{2.10}$$

$$E_{\sigma\sigma'} = \delta_{\sigma\sigma'} \qquad (\sigma, \sigma' = V, \ldots, L) \tag{2.11}$$

$[e_i^k]$ denoting the co-factor of the e_i^k—element of matrix $e_{V-1}(G)$.

3. Let us consider the quadratic form

$$\varphi(a; r) = \sum_{k=1}^{L} a_k q_k^2 = (q, aq) \tag{3.1}$$

where

$$a_{\mu\nu} = \delta_{\mu\nu} \cdot a_\nu \tag{3.2}$$

We have

$$\varphi(a; r) = (r, \Lambda^{-1T} a \Lambda^{-1} r) \tag{3.3}$$

$$\Gamma = \Lambda^{-1T}\, a \Lambda^{-1} \begin{bmatrix} c, & b \\ b^T, & a \end{bmatrix}, \tag{3.4}$$

where a, b, c are matrices with the following elements [6]:

$$a_{\sigma\sigma'} = \sum_{s=1}^{V-1} \sum_{k,k'=1}^{V-1} \alpha_s [e_s^k]\, [e_s^{k'}]\, e_\sigma^k\, e_{\sigma'}^{k'} + \delta_{\sigma\sigma'}\, \alpha_{\sigma'} \quad (\sigma, \sigma' = V, \ldots, L) \tag{3.5}$$

$$b_{r\sigma} = -\sum_{s=1}^{V-1} \sum_{k=1}^{V-1} \alpha_s [e_s^r]\, [e_s^k]\, e_\sigma^k \quad \begin{pmatrix} r = 1, \ \ldots, V-1 \\ \sigma = V, \ldots, L \end{pmatrix} \tag{3.6}$$

$$c_{rr'} = \sum_{s=1}^{V-1} \alpha_s [e_s^r]\, [e_s^{r'}] \quad (r, r' = 1, \ldots, V-1). \tag{3.7}$$

The inverse form can be written in the following way [3]

$$\varphi^{-1}(a; r) = (r, \Gamma^{-1} r), \tag{3.8}$$

where

$$\Gamma^{-1} = \Lambda a^{-1} \Lambda^T = \begin{vmatrix} \varepsilon_{11}, & \ldots, \varepsilon_{1V-1}, & \dfrac{e_V^1}{\alpha_V}, & \ldots, \dfrac{e_L^1}{\alpha_L} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \varepsilon_{V-1,1}, \ldots, \varepsilon_{V-1\,V-1}, & \dfrac{e_V^{V-1}}{\alpha_V}, & \ldots, & \dfrac{e_L^{V-1}}{\alpha_L} \\ \dfrac{e_V^1}{\alpha_V}, & \ldots, \dfrac{e_V^{V-1}}{\alpha_V}, & \dfrac{1}{\alpha_V}, & \ldots, 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dfrac{e_L^1}{\alpha_L}, & \ldots, \dfrac{e_L^{V-1}}{\alpha_L}, & 0, & \ldots, \dfrac{1}{\alpha_L} \end{vmatrix} \tag{3.9}$$

with

$$\varepsilon_{rr'} = \sum_{k=1}^{L} \frac{e_k^r\, e_k^{r'}}{\alpha_k} \quad (r, r' = 1, \ldots, V-1). \tag{3.10}$$

Now let us consider the form

$$\varphi(a; p) = \operatorname*{extr}_{q_V, \ldots, q_L} \varphi(a; r) = [p, (c - b^T a^{-1} b)\, p]. \tag{3.11}$$

The inverse quadratic form is equal [2], [3] to

$$\varphi^{-1}(a; p) = [p, (c - b^T a^{-1} b)^{-1} p] = \varphi^{-1}(a; r)|_{q_V = \ldots = q_L = 0}. \tag{3.12}$$

From (3.9) and (3.10) we obtain

$$\varphi^{-1}(a; p) = \sum_{\sigma=1}^{L} \frac{1}{\alpha_\sigma} \left(\sum_{k=1}^{V-1} e_k^\sigma\, p_\sigma \right)^2 \tag{3.13}$$

and

$$\varphi(\alpha;p) = -\frac{\begin{vmatrix} \varepsilon_{11}, & \ldots, \varepsilon_{1\nu-1}, & p_1 \\ \cdots & \cdots & \cdots \\ \varepsilon_{\nu-1,1}, & \ldots, \varepsilon_{\nu-1\,\nu-1}, & p_{\nu-1} \\ p_1, & \ldots, p_{\nu-1}, & 0 \end{vmatrix}}{\begin{vmatrix} \varepsilon_{11}, & \ldots, \varepsilon_{1\nu-1} \\ \cdots & \cdots \\ \varepsilon_{\nu-1,1}, & \ldots, \varepsilon_{\nu-1\,\nu-1} \end{vmatrix}} . \tag{3.14}$$

This is a positive definite matrix only if the parameters $\alpha_1, \ldots, \alpha_{\nu-1} \geqslant 0$ and if the parameters of independent internal lines $\alpha_\nu, \ldots, \alpha_L > 0$ [6], [7].

We may obtain from formula (3.11) another form for $\varphi(\alpha;p)$ [6], namely

$$\varphi(\alpha;p) = C - (B, A^{-1}B), \tag{3.15}$$

where

$$C = (p, cp), \qquad B = bp, \qquad A = a. \tag{3.16}$$

We can also write

$$\varphi(\alpha;p) = \frac{\begin{vmatrix} A, & B \\ 0, & C - (B, A^{-1}B) \end{vmatrix}}{|A|} = \frac{\begin{vmatrix} A, & B \\ B^T, & C \end{vmatrix}}{|A|} \tag{3.17}$$

and

$$\varphi(\alpha;p) = \frac{|C|}{|A|} \cdot \left| A - \frac{B \cdot B^T}{|C|} \right| . \tag{3.18}$$

4. The diagram's contribution to the scattering amplitude is as follows

$$F_G(p) \lim_{\varepsilon \to +0} \int dq_1 \ldots \int dq_L \prod_{r=1}^{L} \delta\left(\sum_{k=1}^{L} e_k^r q_k - p_r\right) \prod_{s=1}^{L} (q_s^2 - m_s^2 + i\varepsilon)^{-1} \tag{4.1}$$

or, using the α representation [6], [10],

$$F_G(p) = \lim_{\varepsilon \to +0} \pi^{2C} \left(\frac{1}{i}\right)^{L+C} \cdot \delta(\Sigma p) \int_0^\infty d\alpha_1 \ldots \int_0^\infty d\alpha_L M^{-2}(\alpha) \exp i \times \tag{4.2}$$

$$\times \left| q(\alpha;p) - \sum_{k=1}^{L} \alpha_k (m_k^2 - i\varepsilon) \right|,$$

where

$$M(\alpha) = \det A(\alpha) \tag{4.3}$$

and where C denotes the number of independent internal momenta. From the Euler-Poincaré theorem we have

$$C = L - V + 1. \tag{4.4}$$

Let us denote by $R(G)$ the region of $p(p_1, \ldots, p_{\nu-1})$—space which satisfies the inequality

$$\varphi(\alpha; p) - \sum_{k=1}^{L} \alpha_k m_k^2 < 0 \quad \text{for all} \quad \alpha \geqslant 0. \tag{4.5}$$

If $p \in R(G)$, it can be easily shown that the contribution has no singularities [2], [3], [8].

It is important to stress that this definition is not dependent on the transformation properties of the scattered particles [1], [2]. If we take into account, e.g., the spinor particles, we obtain in the formula for contribution the products of polynomials in p of degree 1 although the exponential part of the integrand is not changed.

The renormalization procedure also does not influence $R(G)$, because in this case we have [7]:

$$\tau^2 \cdot \varphi(\alpha; p) - \sum_{k=1}^{L} \alpha_k m_k^2 < 0 \text{ for all } \alpha \geqslant 0 \text{ and } 0 \leqslant \tau \leqslant 1. \tag{4.6}$$

Therefore, $R(G)$ is the minimal region of p satisfying inequality (4.6). To demonstrate the dispersion relations for any process it is sufficient to know the following region:

$$R = \bigcap_{G \in \mathcal{G}} R(G), \tag{4.7}$$

where $\mathcal{G}$ denotes the full set of diagrams of this process.

If inequality $R(G) \subseteq R(G')$ holds, we say that diagram G majorizes diagram G' [3]. This relation will be certainly satisfied if:

$$H_G(p) \geqslant H_{G'}(p) \tag{4.8}$$

where

$$H_G(p) = \max_{\alpha} \left\{ \frac{\varphi(\alpha; p)}{\sum_{k=1}^{L} \alpha_k m_k^2} \right\}^{1/2}. \tag{4.9}$$

It is not difficult to demonstrate that $H_G(p)$ have all properties of norms in the Euclidean space $p(p_1, \ldots, p_{\nu-1})$ [3].

Inequality (4.9) implies the inverse inequality between the norms in the conjugated space $Z(Z_1, \ldots, Z_{\nu-1})$ [3]. The conjugated norm with $H_G(p)$ is determined as follows:

$$H_G^*(Z) = \max_{H_G(p) \leqslant 1} |(Z, p)| \tag{4.10}$$

It is easy to show [3] that the norm $H_G^*(Z)$ can be written in the form

$$H_G^*(Z) = \min_{\alpha} \left\{ \varphi^{-1}(\alpha; Z) \cdot \sum_{k=1}^{L} \alpha_k m_k^2 \right\}^{1/2} \tag{4.11}$$

where $\varphi^{-1}(\alpha; Z)$ is inverse to $\varphi(\alpha; Z)$.

The extremal values of parameters α can be found with the help of the following:

THEOREM 1. *Let $\{H_\alpha(p)\}$ denote the set of norms in Euclidean space p which are continuous functions of α. We denote the set of conjugated norms by $\{H^*_\alpha(Z)\}$.*

If α' and α'' are two elements with properties

$$H_\alpha(p) \leqslant H_{\alpha'}(p) = \max_\alpha \{H_\alpha(p)\} \tag{4.12}$$

and

$$H^*_\alpha(Z) \geqslant H^*_{\alpha''}(Z) = \min_\alpha \{H^*_\alpha(Z)\} \tag{4.13}$$

then $\alpha' = \alpha''$.

Proof. It follows from (4.13) and (4.12) that $H_\alpha(p) \leqslant H_{\alpha''}(p)$ and by comparison with (4.13) we obtain

$$H_{\alpha'}(p) = H_{\alpha''}(p) \quad \text{and, consequently,} \quad \alpha' = \alpha''. \tag{4.14}$$

From (3.13) and (4.11) we have [3]:

$$\alpha'_s = \frac{\sum_{j=1}^{V-1} e^j_s Z_j}{m_s} \cdot \left\{ \frac{\sum_{k=1}^{L} \alpha_k m_k^2}{\sum_{k=1}^{L} \frac{1}{\alpha_k} \left(\sum_{j=1}^{V-1} e^j_k Z_j \right)^2} \right\}^{1/2} \qquad (s = 1, \ldots, L) \tag{4.15}$$

and

$$H^*_{\alpha'}(Z) \equiv H^*_G(Z) = \sum_{k=1}^{L} m_k \left| \sum_{j=1}^{V-1} e^j_k Z_j \right|. \tag{4.16}$$

Using Theorem 1 we get

$$H_{\alpha''}(p) \equiv H_G(p) = \left\{ \frac{\varphi\left(\ldots, \frac{\sum_{j=1}^{V-1} e^j_s p_j}{m_s}, \ldots; p \right)}{\sum_{k=1}^{L} m_k \left| \sum_{j=1}^{V-1} e^j_k p_j \right|} \right\}^{1/2}, \tag{4.17}$$

where $\varphi(\alpha; p)$ is defined by formulae (3.14), (3.15), (3.17), (3.18).

If diagram G possesses internal vertices, we obtain somewhat different formulae for norms [2], [3]. In this case we have namely:

$$H_G(p) = \left\{ \frac{\varphi\left(\ldots, \frac{\sum_{j=1}^{V-1} e^j_s p_j}{m_s}, \ldots; p \right)}{\sum_{k=1}^{L} m_k \left| \sum_{j=1}^{V-1} e^j_k p_j \right|} \right\}^{1/2}_{p \text{ int} = 0} \tag{4.18}$$

and

$$H^*_G(Z) = \operatorname*{Min}_{Z \text{ int}} \sum_{k=1}^{L} m_k \left| \sum_{j=1}^{V-1} e^j_k Z_j \right|. \tag{4.19}$$

The fundamental theorem of the theory of majorization of Feynman diagrams may be expressed in the following form:

THEOREM 2. *The sufficient and necessary condition for majorization of an arbitrary diagram G' by diagram G is the existence of inequalities*

$$H_G(p) \geqslant H_{G'}(p)$$

and

$$H_G^*(Z) \leqslant H_{G'}^*(Z)$$

for all Euclidean p and Z, where $H_G(p)$ and $H_G^*(Z)$ are defined by formulae (4.18) *and* (4.19).

The proof follows evidently from the above considerations.

Below we give some theorems which allow an easy orientation in the majorization problem.

THEOREM 3. *If in diagram G the mass of any line is increased, we obtain a diagram G' which will be majorized by G.*

THEOREM 4. *If we put in diagram G a new line between any two vertices, we obtain diagram G' which will be majorized by G.*

THEOREM 5. *If we put in diagram G a new vertex connected with any of the vertices of G, we obtain diagram G' which will be majorized by G.*

The proofs of the above Theorems are elementary. They are left to the reader.

By multiple applications of both the above mentioned operations we can construct any diagram, assuming as starting point the simplest diagrams [8]. It is obvious, that applications in the inverse sequence allow to reduce any complicated diagram to topologically simplest diagrams. It follows from Theorems 4 and 5 that on every stage of this simplification we shall obtain majorizing diagrams.

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REFERENCES

[1] Y. Nambu, Nuovo Cimento, **6** (1957), 1064.
[2] K. Symanzik, Prog. Theor. Phys., **20** (1958), 690.
[3] A. A. Logunov, L. T. Todorov, N. A. Chernikov, Preprint JINR, (1960).
[4] R. Eden, Phys. Rev., **119** (1960), 1763.
[5] J. M. Maluta, JETP, **4** (1961), 1128.
[6] W. Garczyński, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 467.
[7] — , Doklady ANSSSR, [in press], 1961.
[8] N. N. Bogolubov, O. S. Parasiuk, Acta Mathem., **97** (1957), 227.
[9] P. S. Alexandrov, *Algebraic Topology*, Moscow—Leningrad, 1947.
[10] N. N. Bogulobov, D. V. Shirkov, *Introduction to the theory of quantized fields*, INC., New York, London, 1959.

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Allgemeine Relativitätstheorie zur Zeitdilatation bei beschleunigter geradliniger Bewegung

von

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In der letzten Zeit begegnet man öfters der Frage der Zeitdilatation bei hohen Geschwindigkeiten. Alle mit diesem Problem verbundene Paradoxe wurden ausführlich von Schild [1] geklärt. Er zeigte dabei, dass die Zeitdilatation sowohl bei gleichförmig geradlinigen als auch beschleunigten Bewegungen vollständig mit den Gesetzen der speziellen Relativitätstheorie bestimmt ist. Vom rein geometrischen Standpunkt aus wurde diese Frage von Hlavatý [2] diskutiert.

Obwohl alle Unklarheiten schon beseitigt sind, wollen wir in der vorliegenden kurzen Mitteilung dasselbe Problem noch vom Standpunkt der allgemeinen Relativitätstheorie erörtern, da man in der Literatur [3] auch die Behauptung findet, dass es bei beschleunigten Bewegungen unter gewissen Bedingungen sogar zur Zeitkontraktion kommen kann.

Wir nehmen an, dass ein Teilchen, das sich im inertialen Koordinatensystem mit der Metrik

$$ds^2 = -dx^2 - dy^2 - dz^2 + c^2\,dt^2 \tag{1}$$

in Ruhe an der Stelle $x = y = z = 0$ befand, im Zeitpunkt $t = 0$ in eine beschleunigte geradlinige Bewegung längs der Achse X gesetzt wird. Nach dem Äquivalenzprinzip von Einstein kann man jede Beschleunigung eines Probeteilchens mit einem entsprechend gewählten Gravitationsfeld ersetzen, das mit der Angabe seiner Metrik

$$ds^2 = -A(X)\,dX^2 - dY^2 - dZ^2 + D(X)\,c^2\,dT^2 \tag{2}$$

erklärt ist. Hier bezeichnet c die Lichtgeschwindigkeit und $A(X)$, $D(X)$ zwei näher nicht definierte Funktionen, die für $X \leqq 0$ die Bedingung

$$A(X) = 1, \quad D(X) = 1 \tag{3}$$

erfüllen. Im Bereich $X \leqq 0$ ist daher unser Koordinatensystem (2) auch inertial.

Wir setzen weiter voraus, dass die Masse des Teilchens das äussere, mit Gl. (2) gegebene, Gravitationsfeld nicht beeinflusst. Falls es sich um eine künstlich hervorgerufene Beschleunigung handeln sollte (z. B. durch die Antriebsmotoren einer

kosmischen Photonen-Rakete), so ist unsere Voraussetzung mit der äquivalent, dass die die Beschleunigung bewirkende Ursache vom Bewegungszustand des Teilchens unabhängig ist (d. h. die Motoren arbeiten nach dem vorausfestgesetzten Programm, unabhängig von der Geschwindigkeit und Beschleunigugng der Rakete). Dann bewegt sich das Teilchen (bzw. die Rakete) nach einer geodätischen Linie

$$\frac{d^2 x_\varepsilon}{ds^2} + \left\{ \begin{matrix} \varepsilon \\ \alpha\beta \end{matrix} \right\} \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0 .$$

Bei Behandlung des gegebenen Problems werden wir aber statt dieser Bewegungsgleichungen ihr erstes Integral verwenden, das sie vollkommen vertritt und mit Gl. (2) ausgedrückt ist.

Zuerst bestimmen wir die infinitesimale Entfernung $d\sigma$ zweier Punkte an der Achse X in beiden Koordinatensystemen. In dem inertialen ist sie mit dx identisch:

(4a) $$d\sigma = dx .$$

In dem mit der Metrik (2) erklärten Koordinatensystem ist die Entfernung $d\sigma$ mit der bekannten Formel [4a] gegeben, die in unserem Fall die Form

(4b) $$d\sigma = dX\sqrt{A}$$

einnimmt.

Das infinitesimale Eigenzeitintervall dt im inertialen Koordinatensystem ist identisch mit dem Intervall der Zeitkoordinate dT nur im Bereich $X \leqq 0$. Um das Verhältnis zwischen dt und dT zu bestimmen, müssen wir den Vektor dT vom Punkte $(X, 0, 0)$ längs der Achse X in den Punkt $(0, 0, 0)$ parallel verschieben. Für die Parallelverschiebung eines Vektors ζ^ε gilt die aus der Tensoranalyse im Riemannschen Raum bekannte Formel

$$d\zeta^\varepsilon = -\left\{ \begin{matrix} \varepsilon \\ \alpha\beta \end{matrix} \right\} \zeta^\alpha dx_\beta .$$

Da in unserem Fall

$$\zeta^i = 0, \quad (i = 1, 2, 3); \quad \zeta^4 = dT$$

und von den Christoffelschen Symbolen $\left\{ \begin{matrix} 4 \\ \alpha\beta \end{matrix} \right\}$ nur das folgende von Null verschieden ist

$$\left\{ \begin{matrix} 4 \\ 14 \end{matrix} \right\} = \frac{1}{2D} \frac{dD}{dX} ,$$

nimmt die vorangehende Differentialgleichung die Form

$$d\zeta^4 = -\frac{1}{2D} \frac{dD}{dX} dX\zeta^4$$

ein. Nach der Integration längs der Achse X finden wir

$$\zeta^4 = \zeta_0^4/\sqrt{D} .$$

wobei ζ_0^4 den Vektor ζ^4 an der Stelle bezeichnet, wo $D = 1$. Die gesuchte Formel lautet daher

(5) $$dT = dt/\sqrt{D} .$$

Die relative Geschwindigkeit β im inertialen Koordinatensystem ist mit der Beziehung

$$\beta = \frac{v}{c} = \frac{1}{c}\frac{dx}{dt} \tag{6}$$

erklärt. Nach Einsetzen aus Gln. (4a, b) und (5) ergibt sich

$$\beta = \frac{1}{c}\frac{\sqrt{A}}{\sqrt{D}}\frac{dX}{dT}. \tag{7}$$

Da die Eigenzeit τ des beschleunigten Probeteilchens mit dem Intervall ds nach der bekannten Formel

$$d\tau = ds/c \tag{8}$$

verbunden ist, erhalten wir mit Berücksichtigung der Gln. (8), (5) und (7) aus Gl. (2) die resultierende Beziehung

$$d\tau = dT\sqrt{D}\sqrt{1-\frac{1}{c^2}\frac{A}{D}\left(\frac{dX}{dT}\right)^2} = dt\sqrt{1-\beta^2}. \tag{9}$$

die die Zeitdilatation auch bei beschleunigter geradliniger Bewegung eindeutig festlegt.

In der Newtonschen Annäherung haben wir [4b]

$$A = 1, \qquad D = 1+\frac{2\varphi}{c^2} = 1-\frac{2U}{c^2}. \tag{10}$$

Mit U bezeichnen wir im Einklang mit Fock [3] das negative Newtonsche Gravitationspotential

$$U = -\varphi,$$

das an der Stelle, wo die Geschwindigkeit des Teilchens gleich Null ist oder wird, verschwinden muss, da hier das Koordinatensystem (2) mit dem inertialen identisch wird. Der Erhaltungssatz der Energie lautet daher

$$\frac{1}{2}v^2 - U = 0. \tag{11}$$

Aus dem Ausdruck für das Intervall ds in der Newtonschen Annäherung leitete Fock [3] die Gleichung (62,05) ab, die in unserer Bezeichnungsweise die Form

$$d\tau = dT\left[1-\frac{1}{c^2}\left(\frac{1}{2}v^2+U\right)\right] \tag{12}$$

hat (vgl. auch [4b]). Mit Hilfe der Gl. (11) und bei Berücksichtigung der Tatsache, dass dT nicht mit dt identisch ist, sondern unter ihnen die Beziehung (5) gilt, erhalten wir schliesslich

$$d\tau = dT(1-\beta^2) \cong dt\frac{1-\beta^2}{1-\frac{1}{2}\beta^2} \cong dt\left(1-\frac{1}{2}\beta^2\right), \tag{13}$$

was nichts anders als die Annäherungsform der exakten Gleichung (9) ist.

Wir nehmen nun nach Fock [3] an, dass die Geschwindigkeit v während des Bremsens linear mit der Zeit sinkt

$$v = v_0 - g\,(t - t_1). \tag{14}$$

Nach Einsetzen in Gl. (13) haben wir

$$\int_{\tau_1}^{\tau_2} d\tau = -\int_{v_0}^{0}\left(1 - \frac{1}{2}\,\frac{v^2}{c^2}\right)\frac{dv}{g} = \frac{1}{2}\,t^* - \frac{1}{12}\,\frac{v_0^2}{c^2}\,t^*, \tag{15}$$

wobei wir die von Fock eingeführte Substitution

$$t^* = 2v_0/g \tag{16}$$

verwendet haben.

Nach einer kurzen Zwischenrechnung finden wir, dass der Unterschied in den Zeitangaben einer in Ruhe verharrenden Uhr A und einer anderen Uhr B, die sich während der Zeitspanne t_1 geradlinig mit konstanter Geschwindigkeit v_0 bewegt hat, dann nach Gl. (14) zuerst bis zum Stehen gebracht und darauf wieder auf die Geschwindigkeit $-v_0$ beschleunigt worden ist und zuletzt sich gleichförmig mit dieser Geschwindigkeit $-v_0$ während der neuen Zeitspanne t_1 bewegt hat, mit der Formel

$$\tau_A - \tau_B = \frac{v_0^2}{c^2}\left(t_1 + \frac{1}{6}\,t^*\right) > 0 \tag{17}$$

gegeben ist. Eine Zeitkontraktion, wie Fock [3] sie nachzuweisen versuchte, findet nie statt.

Das Ergebnis von Fock, der denselben Fall berechnete,

$$\tau_A - \tau_B = \frac{v_0^2}{c^2}\left(\frac{1}{2}\,T - \frac{2}{3}\,t^*\right) = \frac{v_0^2}{c^2}\left(t_1 - \frac{1}{6}\,t^*\right) \tag{62, 16}$$

ist nicht richtig: In der ersten Reihe darf man nicht, wie Fock [3] irrtümlicherweise gemacht hat, die Nullstelle des Gravitationspotential U in Gl. (11) beliebig wählen, denn sie ist eindeutig mit der Bedingung festgelegt, dass die Geschwindigkeit des Teilchens an dieser Stelle gleich Null ist oder wird (vgl. Gl. (11)). Der erste Irrtum von Fock liegt daher in seiner Gleichung (62, 07), die bei Berücksichtigung des konstanten Potentials U_0 (dessen Einführung eine nutzlose mathematische Komplikation darstellt, aber keinen wesentlichen physikalischen Sinn hat) wie folgt lauten soll

$$\frac{1}{2}\,v^2 - U = -\,U_0. \tag{18}$$

Schon nach Beseitigung dieses Versehens gibt es keine Zeitkontraktion mehr, nur der Betrag der Zeitdilatation ist noch unrichtig angegeben.

Das zweite Versehen von Fock liegt darin, dass er das infinitesimale Intervall der Zeitkoordinate dT im beschleunigten Koordinatensystem irrtümlicherweise mit der Eigenzeit dt des inertialen Koordinatensystems identifizierte.

Die von Fock berechnete Zeitdifferenz (62,16) gilt für den Fall, dass die Uhr A sich während der oben beschriebenen beschleunigten Bewegung der Uhr B plötzlich mit der konstanten Geschwindigkeit v_0 längs der Achse X bewegen und kein Unterschied zwischen dT und dt bestehen würde.

Zusammenfassung

1. Jede geradlinige beschleunigte Bewegung ist stets mit einer Zeitdilatation verbunden. Eine Zeitkontraktion findet nie statt.

2. Die Gesetze der allgemeinen Relativitätstheorie geben für die Zeitdilatation während einer beschleunigten geradlinigen Bewegung dieselbe Formel an, die sich aus der speziellen Relativitätstheorie ergibt.

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SCHRIFTTUM

[1] A. Schild, Am. Math. Monthly, **66** (1959), 1.

[2] V. Hlavatý, J. Math. Mech., **9** (1960), 733.

[3] V. A. Fock, *Theorie des Raumes, der Zeit und der Gravitation* [russisch], Moskau, 1955, S. 284—287.

[4] L. D. Landau, E. M. Lifšic, *Theorie des Feldes* [russisch] 3. Ausg., Moskau, 1960, [a] — S. 285; [b] — S. 299, Gl. (87, 10).

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Atomic Theory of Light Propagation, Refraction and Reflection in Isotropic Media with Consideration of Electric Quadrupole Radiation

by

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Presented by W. RUBINOWICZ on August 30, 1961

The dispersion formula of Lorentz-Lorenz and also the Fresnel formulae relating to the rerraction and reflection of light have been derived from the molecular theory of light propagation, refraction and reflection in homogeneous isotropic media. The theory is classic statistical and relates to electric dipole radiation, assuming a constant number of identical molecules per unit volume.

On the ground of this theory, the author proposes to derive a quantum-corresponding theory for a homogeneous isotropic medium whose hydrogen-like atoms emit dipole or quadrupole electric radiation. The quantum theory will be applied to a single atom, and the classical theory to the radiation field. Following Oseen [1], atomic interaction by means of the fields will be taken into account, so that the theory will be valid for dense media also. Thus, by applying thereto the Ewald-Oseen theorem (Born-Wolf, [2]), generalized semi-classical formulae, namely, the Lorentz-Lorenz dispersion formula and the Fresnel formulae relating to light refraction and reflection will be derived for a homogeneous isotropic medium whose hydrogen-like atoms emit dipole or quadrupole electric radiation.

1. According to Stratton [3], the electric field radiated by an electric dipole and an electric quadrupole with magnetic dipole, on the assumption of large wavelength of the emitted wave with respect to the dimensions of the oscillator (atom), is given by

$$(1)\qquad \vec{E}' = -\frac{e^{-i\omega t}}{i\omega}\,\mathrm{rot}\,\mathrm{rot}\left\{\left(\int \vec{j}\,d\tau_0\right)\frac{e^{i\frac{\omega R}{c}}}{R} + \left(\int (\vec{r}_0\,\vec{R})\,\vec{j}\,d\tau_0\right)\frac{1}{R}\,\frac{d}{dR}\left(\frac{e^{i\frac{\omega R}{c}}}{R}\right)\right\}.$$

with the following notation:

r_0 — atomic radius,
$d\tau_0$ — space element of atom,
R — distance of point of observation from atom,
ω — frequency of light wave,
$\vec{j}$ — current density.

By Waller [4], the current density in Eq. (1) is assumed as the Schrödinger current whose value is given by the total interaction between the perturbing radiation field and the atom. This interaction will be computed from perturbation theory on the assumption that, in addition to electric-dipole transitions, electric-quadrupole transitions can take place within the atom. In our computations, use will be made of a paper by Rubinowicz [5] for transforming the integral expressions involving the current density. In order to obtain isotropicity, use is made of the selection rules appropriate for each kind of radiation, with summation over the magnetic quantum numbers.

As a result of the foregoing operations, Eq. (1) assumes the form

$$\text{(1a)}\qquad \vec{E}' = -\frac{e^{-i\omega t}}{\omega}\,\mathrm{rot}\,\mathrm{rot}\left\{(\alpha+\alpha_1)\,\vec{E}_0\frac{e^{i\frac{\omega R}{c}}}{R} + \right.$$

$$\left. +i[(\beta+\beta_1)\,\vec{E}_0\,(\vec{k}\vec{R})+\beta\vec{k}\,(\vec{E}_0\,\vec{R})]\frac{1}{R}\,\frac{d}{dR}\left(\frac{e^{i\frac{\omega R}{c}}}{R}\right)\right\},$$

wherein, in addition to the notation of Eq. (1), we have

$\vec{E}_0$ — amplitude of perturbing electric field,

$\vec{k}$ — vector of wave propagation in the medium.

The coefficients $\alpha+\alpha_1$ characterize the electric dipole radiation and are defined as follows:

$$\text{(2)}\qquad \alpha_1 = \frac{e^2}{m\omega},$$

$$\text{(3)}\qquad \alpha = -\frac{e^2}{m\omega}\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2}f_{sn} = -\frac{e^2}{m\omega}\sum_n\sum_i \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2}\sum_m f_{sn},$$

where

$$\text{(4)}\qquad \sum_m f_{sn} = \frac{2m}{3\hbar}\,\omega_{sn}\left\{\left(\int_0^\infty r_0^3\,R_{n,\,l+1}(r_0)\,R_{n'\,l'}(r_0)\,dr_0\right)^2 (l+1)\,\delta_{l+1,\,l'} + \right.$$

$$\left. +\left(\int_0^\infty r_0^3\,R_{n,\,l-1}(r_0)\,R_{n'\,l'}(r_0)\,dr_0\right)^2 l\delta_{l-1,\,l'}\right\}.$$

The coefficient α_1 alone denotes the Thomsonian term of the electric dipole radiation (Waller [4]). The coefficient β relates to electric quadrupole radiation (to the term with quantum transitions), whereas β_1 represents the Thomsonian term of electric quadrupole and magnetic dipole radiation. The coefficients β and β_1 are defined by

$$\text{(5)}\qquad \beta_1 = \frac{e^2}{m\omega}\int_0^\infty r_0^4\,R_{nl}^2(r_0)\,dr_0,$$

$$\text{(6)}\qquad \beta = -\frac{e^2}{m\omega k^2}\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2}g_{sn} = -\frac{e^2}{m\omega k^2}\sum_n\sum_i \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2}\sum_m g_{sn},$$

with

$$(7)\qquad \sum_m g_{sn} = \frac{mk^2\,\omega_{sn}}{10\hbar}\left\{\left(\int_0^\infty r_0^4\, R_{nl}(r_0)\, R_{n'l'}(r_0)\, dr_0\right)^2 \frac{(2l+1)\,l(l+1)}{3\,(2l-1)\,(2l+3)}\,\delta_{ll'} + \right.$$

$$+\left(\int_0^\infty r_0^4\, R_{n,\,l+2}(r_0)\, R_{n'l'}(r_0)\, dr_0\right)^2 \frac{(l+1)\,(l+2)}{2\,(2l+3)}\,\delta_{l+2,\,l'} +$$

$$\left.+\left(\int_0^\infty r_0^4\, R_{n,\,l-2}(r_0)\, R_{n'l'}(r_0)\, dr_0\right)^2 \frac{l\,(l-1)}{2\,(2l-1)}\,\delta_{l-2,\,l'}\right\}.$$

The quantities in Eqs. (2)—(7) denote:

e — elementary charge,

m — electron mass,

f_{sn} — electric dipole oscillator strength,

g_{sn} — electric quadrupole oscillator strength,

$\omega_{sn} = \frac{1}{\hbar}(E_s - E_n)$ — frequency of quantum transition between virtual and initial states of atom.

The virtual state of the atom is given by $\psi_s = R_{nl}(r_0)\, P_l^{|m|}(\cos\vartheta)\, e^{im\varphi}\, N_{lm}$, whereas its initial state — by $\psi_n = R_{n'l'}(r_0)\, P_{l'}^{|m'|}(\cos\vartheta)\, e^{im'\varphi}\, N_{l'm'}$.

2. The total electric field $\vec{E}(\vec{r}, t)$ acting on an atom of the dense, homogeneous isotropic medium, is the sum of the external field $\vec{E}^e(\vec{r}, t)$ and the field resulting from the remaining atoms of the medium:

$$(8)\qquad \vec{E}_{(\vec{r},\,t)} = \vec{E}^e_{(\vec{r},\,t)} + \int_\sigma^\Sigma N\vec{E}'_{(\vec{r}',\,t)}\, dV',$$

with N denoting the number of atoms per unit volume; integration is caried out over the region of the medium limited by the surface Σ, with exclusion of the small sphere of surface σ containing the atom under consideration.

The medium is assumed to consist (statistically) of hydrogen-like (spinless) atoms, each of which is in the same initial state. The atoms are assumed to emit electric dipole or quadrupole radiation under the effect of the perturbing field.

Substituting (1a) into Eq. (8), we get

$$(9)\qquad \vec{E}_{(\vec{r},\,t)} = \vec{E}^e_{(\vec{r},\,t)} - \frac{e^{-i\omega t}}{\omega}\int_\sigma^\Sigma \mathrm{rot}\,\mathrm{rot}\left\{N(\alpha+\alpha_1)\,\vec{E}_0\,\frac{e^{i\left(\vec{k}\vec{r}'+\frac{\omega R}{c}\right)}}{R}\cdot + \right.$$

$$\left. + iN_k\left[(\beta+\beta_1)\,\vec{E}_0(\vec{k}\vec{R}) + \beta\vec{k}\,(\vec{E}_0\,\vec{R})\right]\frac{e^{i\vec{k}\vec{r}'}}{R}\,\frac{d}{dR}\left(\frac{e^{i\frac{\omega R}{c}}}{R}\right)\right\} dV'.$$

where the number of atoms per unit volume, which emit electric dipole radiation, has been denoted by N, and that of the atoms emitting quadrupole radiation — by N_k.

We make use of Green's theorem for transforming the integro-differential equation (9). As a result of this, the volume integrals therein are replaced by integrals

over the surfaces Σ and σ. Computing the surface integrals and applying the Ewald-Oseen extinction theorem into Eq. (9) (as transformed by applying Green's theorem), we obtain the following generalized semi-classical Lorentz-Lorenz formula:

$$(10)\qquad n^2-1=\frac{4\pi}{3}(n^2+2)N\frac{e^2}{m\omega^2}\left(\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2}f_{sn}-1\right)+$$

$$+\frac{4\pi}{5}(2n^2+3)N_k\frac{e^2}{m\omega^2}\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2}g_{sn}-$$

$$-\frac{4\pi}{5}(n^2+4)n^2N_k\frac{e^2}{mc^2}\int_0^\infty r_0^4 R_{nl}^2(r_0)\,dr_0.$$

The first two right-hand terms in Eq. (10) relate to electric dipole radiation, the third — to electric quadrupole radiation (the term with quantum transitions), whereas the fourth corresponds to the Thomsonian term of electric quadrupole and magnetic dipole radiation.

3. The Ewald-Oseen extinction theorem for the medium under consideration is given by the following equation:

$$(11)\qquad \vec{E}^e_{(\vec{r},t)}-\frac{e^{-i\omega t}}{\left(\frac{\omega}{c}\right)^2(n^2-1)}\operatorname{rot}\operatorname{rot}\int_\Sigma\left\{\vec{F}(\vec{r}')\frac{\partial\varphi}{\partial R}-\varphi\frac{\partial\vec{F}(\vec{r}')}{\partial R}+\vec{Q}_1(\vec{r}')\frac{\partial(\vec{k}\nabla)\varphi}{\partial R}-\right.$$

$$\left.-(\vec{k}\nabla)\varphi\frac{\partial\vec{Q}_1(\vec{r}')}{\partial R}+\vec{Q}_2(\vec{r}')\frac{\partial(\vec{E}_0\nabla)\varphi}{\partial R}-(\vec{E}_0\nabla)\varphi\frac{\partial\vec{Q}_2(\vec{r}')}{\partial R}\right\}dS'=0,$$

where

$$\varphi=\frac{e^{i\frac{\omega R}{c}}}{R},$$

$$\vec{F}(\vec{r}')=N\frac{\alpha+\alpha_1}{\omega}\vec{E}_0\,e^{i\vec{k}\vec{r}},$$

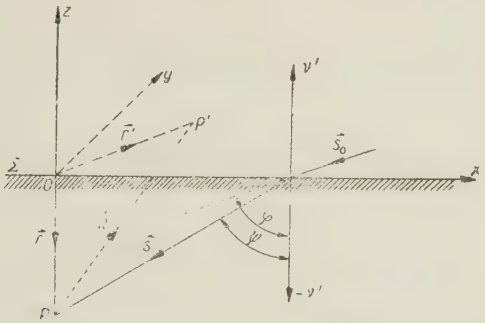
$$\vec{Q}_1(\vec{r}')=iN_k\frac{\beta+\beta_1}{\omega}\vec{E}_0\,e^{i\vec{k}\vec{r}},$$

$$\vec{Q}_2(\vec{r}')=iN_k\frac{\beta}{\omega}\vec{k}\,e^{i\vec{k}\vec{r}}.$$

An appropriate coordinate system is chosen for facilitating the computations. The following notation is used herein:

Coordinates of point of observation P: $\vec{r}=(0,0,-r)$,
Coordinates of current point of plane, P': $\vec{r}'=(x',y',0)$,
Distance between points P and P': $\vec{R}=\vec{r}'-\vec{r}=(x',y',r)$,
Direction of refracted wave (in medium): $\vec{s}=(-\sin\psi,0,-\cos\psi)$,
Direction of incident wave (in vacuum): $\vec{s}_0=(-\sin\varphi,0,-\cos\varphi)$.

The point of observation is chosen distant from the edge of the surface Σ, so that $|\vec{R}|$ is very large with respect to the wavelength.



The integral in Eq. (11) is computed by means of the stationary phase method, basing, in the case of electric dipole radiation, on the solution obtained by Rubinowicz [6] or Born-Wolf [2].

With the denotation:

$\vec{A}$ — amplitude of incident wave,

$\vec{E}_0$ — amplitude of refracted wave,

$\vec{R}$ — amplitude of reflected wave,

we obtain the following generalized Fresnel formulae for

a) the refracted wave

$$(12)\quad A_{\perp} = \frac{\sin(\varphi+\psi)}{\sin\psi\cos\varphi} E_{0\perp} \frac{2\pi}{n^2-1} N \frac{e^2}{m\omega^2}\left(\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} f_{sn} - 1\right) + \\ + \frac{\sin(\varphi+\psi)\cos(\varphi-\psi)}{\sin 2\varphi} E_{0\perp} \frac{4\pi}{n^2-1} N_k \frac{e^2}{m\omega^2} \times \\ \times\left(\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} g_{sn} - k^2 \int_0^{\infty} r_0^4 R_{nl}^2(r_0)\, dr_0\right),$$

$$(13)\quad A_{\parallel} = \frac{\sin(\varphi+\psi)\cos(\varphi-\psi)}{\sin\psi\cos\varphi} E_{0\parallel} \frac{2\pi}{n^2-1} N \frac{e^2}{m\omega^2}\left(\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} f_{sn} - 1\right) + \\ + \frac{\sin(\varphi+\psi)[\cos(\varphi-\psi)+\sin(\varphi-\psi)]}{\sin 2\varphi} E_{0\parallel} \frac{4\pi}{n^2-1} N_k \frac{e^2}{m\omega^2} \sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} g_{sn} - \\ - \frac{\sin(\varphi+\psi)\cos(\varphi-\psi)}{\sin 2\varphi} E_{0\parallel} \frac{4\pi}{n^2-1} N_k \frac{e^2}{m\omega^2} k^2 \int_0^{\infty} r_0^4 R_{nl}^2(r_0)\, dr_0,$$

b) the reflected wave

$$(14)\quad R_{\perp} = -\frac{\sin(\varphi-\psi)}{\sin\psi\cos\varphi} E_{0\perp} \frac{2\pi}{n^2-1} N \frac{e^2}{m\omega^2}\left(\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} f_{sn} - 1\right) + \frac{\sin(\varphi-\psi)\cos(\varphi+\psi)}{\sin 2\varphi} E_{0\perp} \frac{4\pi}{n^2-1} N_k \frac{e^2}{m\omega^2} \times \times\left(\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} g_{sn} - k^2 \int_0^{\infty} r_0^4 R_{nl}^2(r_0)\, dr_0\right),$$

$$(15)\quad R_{\parallel} = \frac{\sin(\varphi-\psi)\cos(\varphi+\psi)}{\sin\psi\cos\varphi} E_{0\parallel} \frac{2\pi}{n^2-1} N \frac{e^2}{m\omega^2}\left(\sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} f_{sn} - 1\right) + \frac{\sin(\varphi-\psi)\,[\cos(\varphi+\psi)-\sin(\varphi+\psi)]}{\sin 2\varphi} E_{0\parallel} \frac{4\pi}{n^2-1} N_k \frac{e^2}{m\omega^2} \sum_s \frac{\omega_{sn}^2}{\omega_{sn}^2-\omega^2} g_{sn} - - \frac{\sin(\varphi-\psi)\cos(\varphi+\psi)}{\sin 2\varphi} E_{0\parallel} \frac{4\pi}{n^2-1} N_k \frac{e^2}{m\omega^2} k^2 \int_0^{\infty} r_0^4 R_{nl}^2(r_0)\, dr_0 .$$

The various terms in Eqs. (12)—(15) relate to the different types of radiation (see explanations to Eq. (10)).

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REFERENCES

[1] C. W. Oseen, Ann. d. Phys., **48** (1915), 1.
[2] M. Born, E. Wolf, *Principles of Optics*, London, 1959.
[3] J. Stratton, *Electromagnetic Theory*, New York, 1941.
[4] I. Waller, Zt. f. Phys., **51** (1928), 213.
[5] A. Rubinowicz, Zt. f. Phys., **61** (1930), 338.
[6] — , Unpublished lectures, 1954.

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EXPERIMENTAL PHYSICS

Absorption Spectra of Eosin in Media of Various Polarity and *pH*

by

A. GUTSZE and H. WALERYŚ

Presented by A. JABŁOŃSKI on September 8, 1961

Investigation by a number of authors [1], [5]—[7] proved fluorescein to exist in different neutral and ionic forms according to the *pH* of the medium; each such form presents its own characteristic absorption and emission spectrum.

According to earlier results obtained by us [3], a change in the polarity of the medium not only produces a shift in the positions of the absorption maxima in agreement with considerations due to Lippert and Moll [10], but also affects the chemical equilibrium constant between the different forms of fluorescein present within the solution. A similar effect was observed when varying the concentration of the dye. Thus, it was felt worth to investigate whether eosin (tetrabromofluorescein) — a closely related molecule — also presents these properties.

Absorption spectra of eosin were investigated

1. in methanol, at different concentrations of the dye,
2. in binary methanol $+CCl_4$ solutions, and
3. in aqueous solutions of NaOH and H_2SO_4.

The measurements were carried out as in [3].

Fig. 1 shows absorption spectra of eosin in methanol at concentrations ranging between 3.1×10^{-10} and 3.1×10^{-8} mole/cm^3. The principal absorption maxima occur therein at $\lambda = 526$ mμ and 258 mμ. The position and shape of the band in the visible region do not change with concentrations of the dye; on the other hand, the height of the maxima decreases to a small extent.

The absorption variations of eosin in binary $CH_3OH+CCl_4$ solutions (Fig. 2) reveal a somewhat different character. As the CCl_4 content in the solution increases, the value of the molar extinction coefficient in the visible region decreases very steeply, vanishing almost completely at 99%.

The shape of the spectrum undergoes a change owing to the change in intensity ratio at the 526 mμ and 490 mμ band maxima. In the ultraviolet, the graphs intersect in two points (called isobestic points). Since eosin (tetrabromofluorescein), according to [6], presents analogously to fluorescein the two neutral quinoid and leuco modifications revealing no absorption in the visible region, it seems highly probable that

the fall in absorption in Figs. 1 and 2 is a result of the transition of a certain number of molecules from the ionic into the neutral form. Thus, in the binary solutions, as we pass from polar methanol to non-polar CCl_4, almost all the eosin ions go

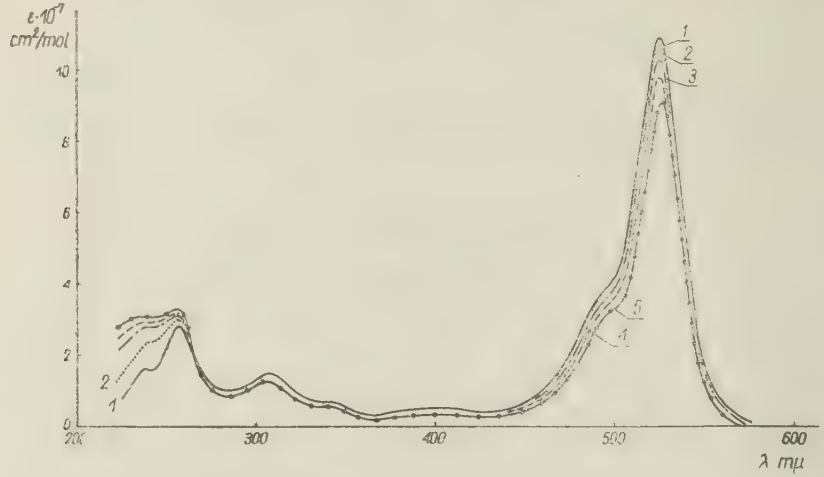

Fig. 1. Absorption of eosin in methanol at concentrations
1 — 3.1×10^{-10}; 2 — 7.8×10^{-10}, 3 — 3.1×10^{-9}; 4 — 7.8×10^{-9}; 5 — 3.1×10^{-8} mole/cm³

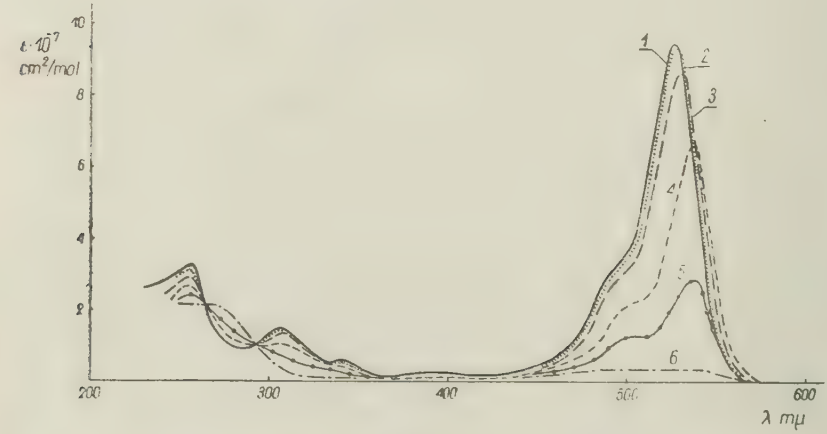

Fig. 2. Eosin absorption at constant concentration $c = 1.5\times10^{-8}$ mole/cm³ in binary solutions of methanol and CCl_4

1 — in pure methanol; 2 — in 24 vols. meth. +1 vol. CCl_4; 3 — 4 vols. meth. +1 vol. CCl_4; 4 — 1 vol. meth. +4 vols. CCl_4; 5 — 1 vol. meth. +24 vols. CCl_4; 6 — 1 vol. meth. +99 vols. CCl_4

over into neutral molecules exhibiting absorption in the ultraviolet region only. On the other hand,thro ughout the concentration variations of eosin in pure methanol investigated by us, only a small number of molecules modify their ionic form,

contrary to what occurs in fluorescein, where the effect is known [3] to be much more marked. This points to some effect of the bromide atoms in the eosin molecule on the stability of its ionic structure. In addition, as polarity decreases, a very marked shift of the maxima of the principal band towards longer wavelengths occurs (Fig.2). Lippert and Moll, and others [11], [12] proved this shift to be intimately related to interaction between the molecular dipole moments of the dye and those of the medium.

Fig. 3 shows the absorption spectra of eosin in media of different basicity. As we proceed from solutions of eosin in methanol towards increasingly basic aqueous solutions, we first find (curves 1—4) a small decrease in absorption accompanied

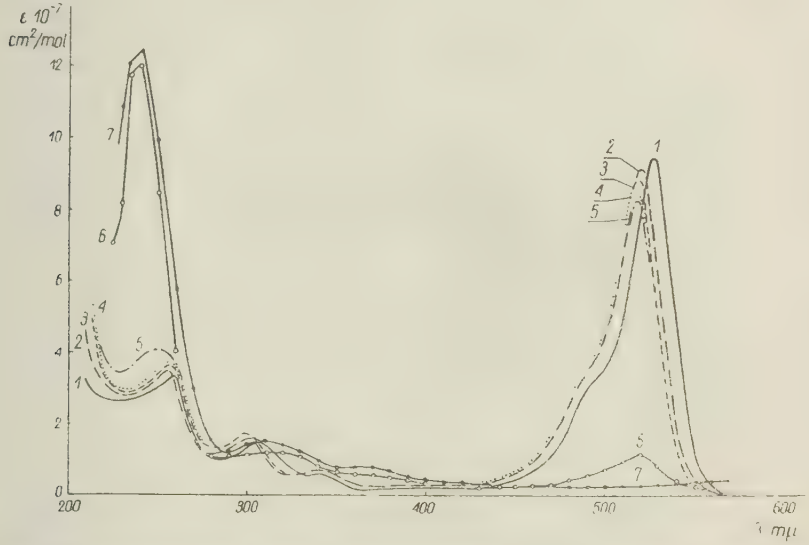

Fig. 3. Eosin absorption at constant concentration c = 1.5×10^{-5} mole/cm³ in basic media 1 — in methanol; 2 — in 1% aqueous solution of methanol; 3 — in 0.001 *n* aqueous solution of NaOH; 4 — in 0.01 *n* aqueous solution of NaOH; 5 — in 0.1 *n* aqueous solution of NaOH; 6 — in 1 *n* aqueous solution of NaOH; 7 — in 10 *n* aqueous solution of NaOH

by a shift in the position of the principal band towards shorter wavelengths, and subsequently (curves 5—7) a very marked fall, almost to zero, of the molar extinction coefficient in the visible part with a rapid increase thereof in the ultraviolet revealing a maximum at $\lambda = 240$ mμ.

Such behaviour of the spectra can be explained by assuming that

1. in solutions of low basicity (curves 1—4), the eosin molecule exists in the form of a univalent anion whose spectrum undergoes a shift towards shorter wavelengths owing to the change in polarity of the medium [10];

2. eosin, like rhodamin, malachite green and other similar dyes [6]—[9], when in a solution of high concentration of OH groups (curves 5—7), attaches this group to the central carbon atom of the dye molecule, the probability for the latter process

being much greater than for the formation of a polyvalent ion [9]. The molecule thus arisen is characterized by new properties, and especially by a loss of absorption in the visible region (and hence a loss of colour) and by very strong absorption in the ultraviolet with a maximum at $\lambda = 240$ mμ (curves 6 and 7 in Fig. 3).

Absorption by eosin in aqueous solutions of sulphuric acid was also investigated. As eosin has a very low solubility in acid media, the spectra were measured for the maximum concentrations of the dye in solution (Fig. 4). The spectrum is seen to undergo far reaching changes throughout the entire region investigated as the acidity of the medium is made to vary. The fact that the molar coefficient of extinction decreases more markedly for the maximum of the 525 mμ band than for that of

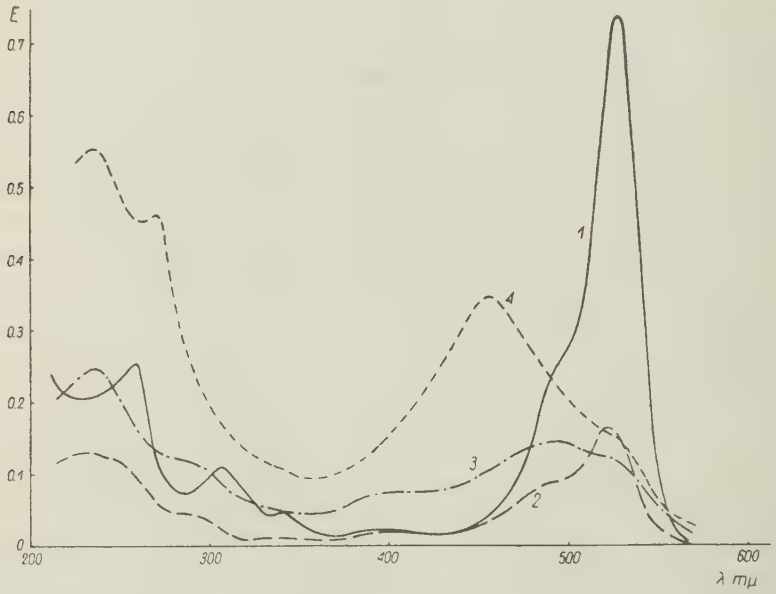

Fig. 4. Eosin absorption in acid medium

1 — in methanol; 2 — in 0.001 *n* aqueous solution of H_2SO_4; 3 — in 0.1 *n* aqueous solution of H_2SO_4; 4 — in 10 *n* aqueous solution of H_2SO_4

the 490 mμ band with increasing acidity of the medium points decidedly to the disappearance of the univalent anion. Already in a 0.1 *n* H_2SO_4 solution, the equilibrium shifts, favouring the cation, to which the 490 mμ band is considered to belong. A further decrease in the *pH* value fails to produce any (relative) increase in the number of cations, but leads to a new molecule possessing a different electronic structure, in accordance with [6], as revealed by the presence of a new, wide and rather intense band with an absorption maximum at $\lambda = 455$ mμ and by a general increase in absorption in the farther ultraviolet. The authors regret not to have been able to determine the values of the maximum concentrations of eosin in the acidic solutions and, thus, not to be in a position to say anything on the role played by the neutral molecules in the process of conversion of anions into cations. It is

noteworthy that, in the basic solutions, too, the univalent anion is accompanied by a univalent cation, albeit in small quantities, the amount of the latter diminishing with increasing *pH* of the solution.

From the foregoing considerations, eosin can be said to be present in the form of cation, neutral molecule, and univalent anion, the latter being the most abundant in neutral solutions and in ones of low polarity. The equilibrium between these forms is highly sensitive with respect to variations in the polarity of the medium and, to a lesser degree, with respect to variations in the dye concentration. This is also true of weakly basic and weakly acidic solutions. In media presenting a high concentration of the OH group, blending of the dye molecule occurs owing to bonding of an OH group to the central carbon atom, whereas in strong acids the absorption spectrum undergoes a complete change pointing to the formation of a new molecule.

The authors wish to thank Professor A. Jabłoński for his valuable suggestions in connection with the present investigation.

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REFERENCES

[1] V. Zanker, W. Peter, Chem. Ber., **1** (1958), 572.
[2] E. W. Grigorian, L. T. Kantardjian, S. S. Chirkinian, Izv. AN. SSSR, **24** (1960), 771.
[3] A. Gutsze, H. Waleryś, Bull. Acad. Polon. Sci., Sér. sci. math., astr. et phys., **9** (1961), 507.
[4] I. S. Yoffe, V. F. Otten, Zhur. Ob. Khim., **31** (1961), 1511.
[5] M. Rozwadowski, Acta Phys. Polon., in press.
[6] W. R. Orndorff, E. F. Hitch, J. Am. Chem. Soc., **36** (1914), 680.
[7] W. R. Orndorff, S. A. Mahood, ibid., **40** (1918), 937.
[8] W. R. Orndorff, W. A. Adamson, ibid., **40** (1918), 1235.
[9] A Thiel, L. Jungfer, Z. anorg. allg. Chem., **49** (1929), 178.
[10] E. Lippert, F. Moll, Z. Electrochem. Ber. Bunsenges. phys. Chem., **58** (1954), 718.
[11] J. Czekalla, Chimia, **15** (1961), 26.
[12] J. E. Platt, J. Chem. Phys., **25** (1956), 80.

БЮЛЛЕТЕНЬ
ПОЛЬСКОЙ АКАДЕМИИ НАУК
СЕРИЯ МАТЕМАТИЧЕСКИХ, АСТРОНОМИЧЕСКИХ
И ФИЗИЧЕСКИХ НАУК

1961

ТОМ IX Резюме статей ВЫПУСК 11

Ю. МУСЕЛЯК, **О НЕКОТОРЫХ ПРОСТРАНСТВАХ ФУНКЦИЙ И ДИСТРИБУЦИЙ III. ПРОСТРАНСТВО *M*-ИНТЕГРИРУЕМЫХ ВЕКТОРНЫХ ДИСТРИБУЦИЙ**

стр. 765—767

В настоящей статье рассматривается пространство $\mathcal{D}'_M(E) = \mathcal{L}_\varepsilon(\mathcal{D}_{Nc}; E)$ векторных дистрибуций и пространство $\overline{\mathcal{D}}'_M = \mathcal{L}_\varepsilon(\mathcal{D}_N; E)$, где E — произвольное локально выпуклое линейное топологическое гаусдорфово пространство. Пространства $\mathcal{D}_N$ функций и $\mathcal{D}'_M$ дистрибуций определены в [3], пространство $\mathcal{L}_\varepsilon(A'_c; B)$ в [7]. Доказывается, что $\overline{\mathcal{D}}'_M(E)$ содержит пространство $\mathcal{L}^*_M[E]$ M-интегрируемых векторных функций и топология в $\mathcal{L}^*_M[E]$ сильнее, чем в $\overline{\mathcal{D}}'_M[E]$.

Р. ТАБЕРСКИЙ, **ДОПОЛНИТЕЛЬНЫЕ СВЕДЕНИЯ ПО (K, φ)-СУММИРУЕМОСТИ**

стр. 769—774

В работе указаны соотношения между сильной $|C^q, 1|$-суммируемостью и (K, φ)-суммируемостью числовых и ортогональных рядов, сформулировано замечание о (K, φ)-суммировании рядов, составленных из производных членов ряда Фурье и установлены асимптотические формулы для некоторых тригонометрических операторов.

М. АЛЬТМАН, **ИТЕРАЦИОННЫЙ МЕТОД ДЛЯ СОБСТВЕННОЙ ПРОБЛЕМЫ ЛИНЕЙНЫХ ОГРАНИЧЕННЫХ ОПЕРАТОРОВ И МАТРИЦ** . . . . .

стр. 775—778

В работе предлагается некоторый итерационный метод нахождения собственных чисел и собственных векторов линейных ограниченных и самосопряженных операторов, действующих в гильбертовом пространстве. Дается также применение этого метода к вычислению собственных чисел и собственных векторов симметричных матриц. Рассматривается также обобщение этого метода, полученное путем введения числового параметра.

М. АЛЬТМАН, **ДВА ИТЕРАЦИОННЫХ МЕТОДА РЕШЕНИЯ ЛИНЕЙНЫХ АЛГЕБРАИЧЕСКИХ УРАВНЕНИЙ** . . . . . стр. 779—783

Вводится понятие знак-градиента, т.е. вектора со знаками градиента, в качестве координат. Для получения следующего приближения передвигаемся вдоль знак-градиента. Таким образом полученный метод требует всего n умножений и одного деления для каждого итерационного шага, не считая сложений и вычитаний. Здесь n является порядком матрицы.

М. АЛЬТМАН, **ОБЩИЙ ПРИНЦИП ДЕФЛЯЦИИ ДЛЯ ЛИНЕЙНЫХ ИТЕРАЦИЙ** стр. 785—790

Работа содержит общий прием сведения итерационных процессов решения линейных уравнений в пространстве Гильберта к подпространству. Это вложение процесса в подпространство и называется дефляцией. Она осуществляется некоторой линейной трансформацией, для которой отношение Райлея на подпространстве совпадает с этим же отношением для оператора уравнения. Таким образом улучшается обусловленность оператора. Некоторые методы раньше предложенные автором являются методами с дефляцией.

М. АЛЬТМАН, **МЕТОД ТИПА НЬЮТОНА ДЛЯ ЛИНЕЙНЫХ УРАВНЕНИЙ** стр. 791—794

В работе предложен итерационный метод решения линейных уравнений в пространстве Гильберта. Этот метод является методом типа Ньютона в смысле данного автором обобщения для функционалов классического метода Ньютона. Особенностью предлагаемого метода является то, что он определяется только одной итерационной формулой, в то время как другие градиентные итерационные методы определяются двумя такими формулами: одна — для приближенного решения и другая (вспомогательная) — для градиента. Это обстоятельство существенно для программирования метода для цифровых вычислительных машин.

ЯН Н. СНЕДДОН, **ЗАМЕТКА О НЕКОТОРЫХ СООТНОШЕНИЯХ МЕЖДУ ТРАНСФОРМАТАМИ ФУРЬЕ И ГАНКЕЛЯ** . . . . . стр. 799—806

В работе рассматриваются зависимости между некоторыми суперпозициями косинусоидальной и синусоидальной трансформаций Фурье и трансформации Ганкеля нулевого и первого порядка. Автором вводятся вспомогательные интегральные трансформации.

Полученные тождества применяются затем для определения зависимости между функциями при предположении, что некоторые их трансформации выполняют данные зависимости.

Автор использует полученные результаты для решения краевых задач в теории потенциала Лапласа для дву- и трех-мерного случаев.

Приводится также зависимость между решениями некоторых таких задач.

Р. Д. АНДЕРСОН и Е. Х. КОННЕЛЛЬ, **ЗАМЕТКА О ЗАМЫКАНИЯХ ОБРАЗОВ ДЕЙСТВИТЕЛЬНЫХ ЧИСЕЛ** . . . . . стр. 807—810

В работе рассматриваются вопросы, касающиеся размерности замыканий образов действительных чисел и вообще одномерных множеств при непрерывных отображениях.

Главные результаты работы содержатся в следующих теоремах:

Теорема 1. *Пусть* $f: R \to T$ *— отображение действительной прямой* R *на* тор T, *данное формулой* $f(x) = (x, \alpha x)$ (мод 1), *где* α *является положительным иррациональным числом.*

Пусть $F: R \times I \to T \times I$ *определяется формулой* $F(r, t) = (f(r), t)$.

Тогда существует неограниченное замкнутое связное множество $C < (R \times I)$ *такое, что замыкание* $(F(C))$ *имеет размерность* $= 1$.

Теорема 2. *Существует непрерывная функция* $f: R \to S$ (*где* S *обозначает единичный квадрат*) *такая, что если* $F: R \times I \to S \times I$ *определено формулой* $F(r, t) = (f(r), t)$ *и если* C *является неограниченным связным подмножеством множества* $R \times I$, *то замыкание множества* $F(C)$ *имеет размерность* > 2.

Теорема 3. *Пусть* R *— полупрямая* $[0, \infty)$ *и пусть* f *— отображение* R *в* S (*где* S *обозначает единичный квадрат в плоскости* xy). *Пусть* F *— отображение* $R \times I^2$ *в* $S \times I^2$ *такое, что* $F(r, p) = (f(r), p)$.

Тогда существует неограниченное, замкнутое, связное множество C *в* $R \times I^2$ *такое, что замыкание* $F(C)$ *имеет размерность не больше* 1.

Д. САДОВСКАЯ, **ЗАДАЧА КОШИ ДЛЯ ПАРАБОЛИЧЕСКОЙ СИСТЕМЫ С КОЭФФИЦИЕНТАМИ ЗАВИСИМЫМИ ОТ НЕИЗВЕСТНЫХ ФУНКЦИЙ** . . стр. 811—816

В настоящей работе автор исследует задачу определения системы функций $u = [u, (X, t), ..., u_N(X, t)]$, удовлетворяющей параболической системе уравнений (1) и выполняющей начальное условие

$$\lim_{t \to 0} u_\alpha(X, t) = f_\alpha(X), \qquad (\alpha = 1, ..., N).$$

Автор предполагает, что коэффициенты системы удовлетворяют условиям Гельдера (4), (5) и уравнению Петровского (6). Заданные непрерывные функции $f_\alpha(X)$ выполняют неравенство (11).

Применяя теорему Шаудера о неподвижной точке преобразования, автор доказывает существование одного, по крайней мере, решения задачи, если заданные постоянные a_f или T достаточно малые.

В. ГАРЧИНСКИЙ, **О ПРОБЛЕМЕ МАЖОРИРОВАНИЯ В ТЕОРИИ ВОЗМУЩЕНИЙ** стр. 819—825

На основании общего параметрического представления для амплитуды рассеяния в теории возмущений рассматривается проблема мажорирования диаграмм Фейнманна.

Установлен новый критерий мажорирования диаграмм.

Доказано несколько теорем, при помощи которых можно легко построить примитивные диаграммы любого процесса.

Я. ПАХНЕР, **ПРИМЕНЕНИЕ ОБЩЕЙ ТЕОРИИ ОТНОСИТЕЛЬНОСТИ К ВОПРОСУ ДИЛАТАЦИИ ВРЕМЕНИ ПРИ УСКОРЕННОМ ПРЯМОЛИНЕЙНОМ ДВИЖЕНИИ** . . . . . стр. 827—831

В работе рассматривается проблема дилатации времени с точки зрения общей теории относительности. Рассматривается именно движение частицы во внешнем однородном гравитационном поле (согласно общей теории относительности гравитационное поле замещает ускорение частицы). Предполагается притом, что причина, вызывающая ускорение частицы, не зависит от состояния движения частицы.

Если, напр., рассматривать движение космической фотонной ракеты, то упомянутое условие сводится к тому, что ее движущие моторы работают по установленной заранее программе, независимо от ее скорости и ускорения.

В работе доказывается, что собственное время частицы приведенной в произвольное прямолинейное ускоренное движение протекает всегда медленнее, чем собственное время наблюдателя, находящегося в состоянии покоя.

Не имеется сокращение времени, что пытался доказать Фок.

А. ВЕЖБИЦКИЙ, **АТОМНАЯ ТЕОРИЯ РАСПРОСТРАНЕНИЯ, ПРЕЛОМЛЕНИЯ И ОТРАЖЕНИЯ СВЕТА В ИЗОТРОПНЫХ СРЕДАХ С УЧЕТОМ КВАДРУПОЛЬНОГО ИЗЛУЧЕНИЯ** . . . . . стр. 833—838

Выведены обобщенные формулы, а именно: дисперсионная Лорентца-Лоренца и касающиеся преломления и отражения света формулы Френеля. Эти формулы были получены на основании квантово-соответствующей теории распространения, преломления и отражения света в изотропных однородных средах. Эта теория была построена на основании классической теории Осеена (1915). В качестве вышеупомянутой изотропной среды принимается среда, составленная из водородоподобных атомов (не учитывая спина), которые могут под влиянием возмущающего поля, эмиттировать электрическое дипольное или квадрупольное излучение.

А. ГУТШЕ и Г. ВАЛЕРЫСЬ, **СПЕКТРЫ ПОГЛОЩЕНИЯ ЭОЗИНА В СРЕДАХ С РАЗНЫМИ ПОЛЯРНОСТЬЮ И** *pH* . . . . . .стр. 839—843

Исследовались спектры поглощения эозина в средах с разной полярностью и различным *pH*. Констатировано, что эозин в нейтральных и малополярных средах может выступать, аналогично как флуоросцеин, в нейтральной форме (абсорбирующей только в ультрафиолете) и в ионной форме, причем изменение концентрации красителя, а в особенности полярности среды, изменяет очень сильно постоянную химического равновесия между этими формами. Перемещение положения максимумов поглощения в видимом пространстве (в видимой части спектра), вызванное этими изменениями, совпадает с результатами, полученными Липпертом и Моллем.

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